On Leader Selection for Performance and Controllability of Multi-Agent Systems

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Outline

- Leader-follower systems
- Problem formulation: Leader selection for joint performance and controllability
- Submodular optimization approach
- Leader selection algorithms and optimality
- Control of gene regulatory networks
- Conclusions
Leader-Follower Systems

- Consist of networked leader and follower nodes
- **Leader nodes**: Act as control inputs to the remaining nodes
  - Broadcast state values to influence/steer follower nodes
  - May be controlled by external operators
- **Follower nodes**:
  - Receive inputs from leaders and neighboring followers
  - Update their internal states based on distributed protocol
  - Broadcast state values to neighbors
- **Question**: How to select nodes to act as leaders?
Related Work

- Efficient combinatorial algorithms for controllability
  - Necessary and sufficient spectral conditions [Tanner `04]
  - Graph-theoretic necessary conditions [Rahmani et al `09]
  - Matching-based algorithm [Liu et al `11, Chapman et al `13, Ruths et al `14]

- Optimization approaches to maximizing performance
  - Supermodular optimization for minimizing link noise and convergence error [Clark et al `14]
  - Convex optimization for minimizing noise error [Fardad et al `11, Lin et al `13]
  - Information centrality-based approach [Fitch et al `13]
Problems Studied

- **First Problem**: Select a set $S$ of up to $k$ leaders to minimize a cost metric $f(S)$ (e.g., error due to noise) while satisfying controllability.

- **Second Problem**: Select a set of up to $k$ leaders based on joint consideration of cost metric and controllability.
System Model

- Network of n agents $V = \{1, \ldots, n\}$, with edge set $E$
- $N(i) := \text{neighbors of node } i$
- Assume that graph is strongly connected
- Agent $i$ maintains time-varying real-valued state $x_i(t)$
- **Follower** agents have state dynamics

$$\dot{x}_i(t) = \sum_{j \in N(i)} W_{ij} x_j(t) + W_{ii} x_i(t)$$

- The states of the set of leader agents $S \subseteq V$ act as control inputs
Controllability

- Network is controllable from leader set $S$ if it can be driven from any initial state to any desired state via inputs from the leader nodes within finite time $T$.
Conditions for Controllability

First Condition: Every **follower** must have at least one path from a **leader**

Second Condition: For every set of followers \( A \), \( |N(A)| \geq |A| \), where \( N(A) \) is the (directed) neighbor set of \( A \)
Graph Controllability Index (GCI)

We define controllability metric (GCI):
\[ c(S) = \text{size of largest subgraph controllable, including leader set } S \]

\[ c(S) = \text{number of leader nodes} + \text{number of controllable follower nodes} \]

\[ |S| \]

\[ g(S) \]
Problem Formulation

- Based on GCI metric $c(S)$, leader selection for joint performance and controllability formulated as

\[
\text{minimize} \quad f(S) - \lambda c(S) \\
\text{s.t.} \quad |S| \leq k
\]

$\lambda \geq 0$ trade-off parameter

- **Goal**: Characterize the structure of GCI as a function of the leader set
Submodularity

A function $f: 2^V \to \mathbb{R}$ is **submodular** if for any sets $S$ and $T$ with $S \subseteq T$ and any element $v \notin T$,

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$$

Function $f$ is **supermodular** if $-f$ is submodular.

**Example**: Set cover

$V = \{1, 2, 3, 4\}$

$f(S) =$ area of discs indexed in $S$

$v = 2, S = \{1\}, T = \{1, 3, 4\}$
Submodularity

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- Example: Set cover

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Matching Interpretation of GCI

- Intuition: $|N(A)| \geq |A|$ for any set of followers $A$ ⇔ each node can control at most one of its neighbors

- Construct bipartite graph with an edge from each node to the neighboring follower it controls
Matching Interpretation of GCI

- Network is controllable if there exists a matching in which each follower node is matched (perfect matching)
  - Each node is controlled by one of its neighbors
- Unmatched nodes of network are not controlled
- Number of controlled follower nodes $g(S) =$ number of matched follower nodes
Submodularity of GCI

- Idea of proof: Adding new node \( v \) to leader set \( S \) increases GCI iff \( v \) is unmatched
  - If \( v \) is matched, then \( v \) is already controllable
- If \( v \) is unmatched with leader set \( T \), then \( v \) is also unmatched when the leader set is \( S \subseteq T \)
  - If adding \( v \) increases \( c(T) \), then adding \( v \) increases \( c(S) \) for \( S \subseteq T \)

\[ c(S \cup \{n_6\}) - c(S) = 1 \]
Submodularity of GCI

- Need to show that, for any \( S \subseteq T \) and \( v \not\in T \),
  \[
  c(S \cup \{v\}) - c(S) \geq c(T \cup \{v\}) - c(T)
  \]
- Two cases for \( c(T \cup \{v\}) - c(T) \)
- First case: \( v \) is unmatched (\( c(T \cup \{v\}) - c(T) = 1 \))
- Second case: \( v \) is matched (\( c(T \cup \{v\}) - c(T) = 0 \))
- First case follows from intuition of previous slide
- In the second case, since \( c(S \cup \{v\}) - c(S) \in \{0, 1\} \), inequality automatically satisfied
- Hence \( c(S) \) is submodular
Algorithm and Optimality Guarantees

- **Greedy algorithm:**
  - Initialize $S = \emptyset$; at each iteration, add leader node that minimizes $f(S) - \lambda c(S)$
  - Terminate after $k$ iterations
- **General optimality result:** Leader set selected by greedy algorithm results in objective value within $(1 - 1/e)$ factor of optimum when $f(S)$ is supermodular
- **Special case:** If $f(S) = 0$ (selection based on controllability alone), chosen leader set is **optimal**
  - Key step in proof: $c(S)$ is a matroid rank function
Controlling the Cell

Development of a disease by modification of the attractor landscape
Curing a disease by regulating the control kernel for normal attractor

Cell fates are steady states (attractors) of gene regulatory network dynamics

Examples: Differentiation of stem cell into blood or neuron
Cellular dynamics are nonlinear (Boolean)

How to ensure that a desired attractor is reached?
Control of Cell Dynamics

- **Input Genes**
  \[ S_1 = \{ c_1, c_2, ..., c_k, f_1, f_2, f_{n-k} \} \]

- **Boolean dynamics**

- **Gene state space** \( \{0,1\}^n \)

- **Approach:** Fix a subset of input genes to the desired attractor state.

- **Remaining genes** reach the desired attractor state via Boolean interactions.

- **How to select a minimum-size set of input genes?**
Selecting Input Genes

- **Current approach**: Select input genes $S$ using a genetic algorithm [Kim et al. '13]
- Fitness function determined by size of input set, fraction $B$ of initial states that converge to desired attractor
  
  $$f(S) = \begin{cases} 
  B^3(n - |S|)^2, & B \neq 1 \\
  2B^3(n - |S|)^2, & B = 1 
  \end{cases}$$

- Claim of [Kim et al. '13]: Control Kernel $S$ selected by algorithm guarantees convergence to desired attractor from any initial state
- We have found a set of initial states that are counterexamples to this claim
- **Analytical approach** to input or control kernel selection, analogous to controllability, needed for this class of networks
Conclusions

- Studied leader selection for joint performance and controllability in leader-follower systems
- Introduced a graph controllability index (GCI) that quantifies level of controllability from leader set
- Proved submodular structure of GCI, leading to efficient leader selection algorithms with provable optimality guarantees
- New methodologies needed for networks with Boolean dynamics
Questions?


