Min-Max Game Problem for Elastic and Visco-elastic Fluid Structure Interactions

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The Abstract Min-Max Game Theory:

- Classic Game Theory: von Neumann.
- Min-Max Game Theory in Dynamics: For objective function \( F(y; u, w) \), we consider the following problem:

\[
\sup_{w} \inf_{u} F(y; u, w)
\]  

(1)

- \( y \) - The dynamical trajectory.
- \( u \) - The control term.
- \( w \) - The disturbance term.
Interest:

- The uniqueness and existence of the feedback solution (a saddle point) \((u^*(y), w^*(y))\) to Problem (1).
  Technique: Calculus of Variations.
- The construction of \((u^*(y), w^*(y))\) that steers the system into optimal trajectory satisfying Problem (1) based on the initial condition state \((y\ value)\) of the system.
  Tools: Non-Standard Riccati Equations.
Literature:


R. Triggiani and J.Z., Min-max game theory and non-standard differential Riccati equations under singular estimates for $e^{At}B$ and $e^{At}G$ in the absence of analyticity, *Set-Valued and Variational Analysis*, 17 (2009), 245–283.

Coupled System and Singular Estimate

Abstract Dynamical System:

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + Bu(t) + Gw(t) \in [D(A^*)]' \\
y(s) &= y_0 \in Y
\end{align*}
\]  \hspace{1cm} (2)

\(u(t) \in L_2([s, T]; U), \ w(t) \in L_2([s, T]; V)\)

\(A, B, G - \) Linear operators such that \(A^{-1}B \in \mathcal{L}(U, Y)\) and \(A^{-1}G \in \mathcal{L}(V, Y)\).

\[J(u, w) = J(u, w, y(u, w))\]

\[= \int_s^T \| Ry(t) \|^2_Z + \| u(t) \|^2_U - \gamma^2 \| w(t) \|^2_V \, dt\]  \hspace{1cm} (3)

Min-Max Game Theory for (2):

\[
\sup_u \inf_w J(u, w)
\]  \hspace{1cm} (4)
\[ \| e^{At}B \|_{\mathcal{L}(U;Y)} = \| B^* e^{A^*t} \|_{\mathcal{L}(Y;U)} \leq \frac{C_T}{t^\alpha}, \quad \forall \ s < t \leq T \]

\[ \| e^{At}G \|_{\mathcal{L}(V;Y)} = \| G^* e^{A^*t} \|_{\mathcal{L}(Y;V)} \leq \frac{C_T}{t^\alpha}, \quad \forall \ s < t \leq T \]

**Remarks:**

- Singular estimates arise from
  1. coupled PDE models combining hyperbolic and parabolic effects (hyperbolic and parabolic component).
  2. unbounded input operators \((B\text{ or }G)\).
- Singular estimates automatically hold for \(A\) analytic.
- \(\alpha\) affects the stability of the system \((2)\) in the long run.
- \(\alpha\) is critical in establishing the wellposedness of the associated nonstandard Riccati equation.
Literature:


Theorem (T. Triggiani and J.Z.): There exists a critical value \( \gamma_c > 0 \),

- If \( 0 < \gamma < \gamma_c \), there is no finite solution for (4) for any \( y_0 \in Y \), since it leads to \( +\infty \) as \( w \to +\infty \).

- If \( \gamma > \gamma_c \), then
  (i) There exists a unique solution pair \( \{ u^*(\cdot; y_0), y^*(\cdot; y_0), w^*(\cdot; y_0) \} \) and the correspondent cost functional \( J(u^*, w^*) \) is the unique solution to (4).

(ii) There exists a bounded non-negative self-adjoint operator \( P(t) = P^*(t) \in \mathcal{L}(Y), \ s \leq t \leq T \), such that:

\[
P(t) \text{ continuous : } Y \to C([s, T]; Y)
\]

(iii) The following pointwise feedback relations hold true:

\[
u^*(t; x) = -B^*P(t)y^*(t; x) \in C([s, T]; U), \quad x \in Y
\]

\[
w^*(t; x) = \gamma^{-2}G^*P(t)y^*(t; x) \in C([s, T]; V), \quad x \in Y
\]
Theorem (continue...):

(iv) $P(t)$ defines the cost functional in (3) for the solution of the min-max game problem initiating at the point $x \in Y$ and at the time $t$, over the interval $[t, T]$, for all $t \in [s, T]$:

$$(P(t)x, x)_Y \equiv \sup_{w \in L_2(t, T; V)} \inf_{u \in L_2(t, T; U)} J(u, w; x)$$

(v) $P(t)$ can be obtained by solving the following non-standard differential Riccati equation, for all $x, y \in D(A)$

$$\begin{cases} 
(P(t)x, y)_Y &= -(Rx, Ry)_Y - (P(t)x, Ay)_Y - (P(t)Ax, y)_Y \\
&\quad+ (B^*P(t)x, B^*P(t)y)_U - \gamma^{-2}(G^*P(t)x, G^*P(t)y)_V \\
\end{cases}$$

$$P(T) = 0$$
The fluid-structure interaction model consists of an elastic solid, immersed in a fluid, with coupling taking place at the interface between the two media. The control acts on the interface and the disturbances are distributed in both the fluid and the solid.
Mathematical Model

\[
\begin{align*}
    u_t - \Delta u + Lu + \nabla p &= w_1 & \text{in} \quad Q_f \equiv \Omega_f \times (0, T] \\
    \text{div} \, u &= 0 & \text{in} \quad Q_f \equiv \Omega_f \times (0, T] \\
    v_{tt} - \text{div} \, \sigma(v) - \rho \text{div} \, \sigma(v_t) &= w_2 & \text{in} \quad Q_s \equiv \Omega_s \times (0, T] \\
    v_t &= u + g_0 & \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0, T] \\
    u &= 0 & \text{in} \quad \Sigma_f \equiv \Gamma_f \times (0, T] \\
    \sigma(v + \rho v_t) \cdot \nu &= \epsilon(u) \cdot \nu - \rho \nu - g_1 & \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0, T] \\
    u(0, \cdot) &= u_0 & \text{in} \quad \Omega_f \\
    v(0, \cdot) &= v_0, \quad v_t(0, \cdot) = v_1 & \text{in} \quad \Omega_s
\end{align*}
\]
$\Omega_s$ - Solid, $\Omega_f$ - Fluid.
• Energy Space: $\mathcal{H} \equiv H \times (H^1(\Omega_s))^n \times (L_2(\Omega_s))^n$ for $\{u, v, v_t\}$

$$H \equiv \{u \in (L_2(\Omega_f))^n : \text{div } u = 0, \quad u \cdot \nu|_{\Gamma_f} = 0\}$$

• Control Space: $\mathcal{U} \equiv (L_2(\Gamma_s))^n$

• Disturbance Space: $\mathcal{V} \equiv (L_2(\Omega_f))^n \times (L_2(\Omega_s))^n \times (L_2(\Gamma_s))^n$
Cost Functional:

\[ J(u, v, g, w) = \int_0^T \left( |g(t)|_U^2 + |R_1 u(t)|_{L^2(\Omega_f)}^2 + |R_2 v(t)|_{H^1(\Omega_s)}^2 + |R_3 v_t(t)|_{L^2(\Omega_s)}^2 \right) dt \]

\[ - \gamma^2 \int_0^T |w(t)|_V^2 dt \]

where \( g = (g_0, g_1) \) and \( w = (w_1, w_2) \);
\( R_1 \in \mathcal{L}(L^2(\Omega_f)), R_2 \in \mathcal{L}(L^2(H^1(\Omega_s))), R_3 \in \mathcal{L}(L^2(\Omega_s)) \)

Min-Max Game Problem:

\[ \sup_{w \in L^2(V)} \inf_{g \in L^2(U)} J(u, v, g, w) \quad (6) \]
Special Case of Fluid-Structure Interaction Model

- Dismatch of the parabolic and hyperbolic component on $\Gamma_s$.

Theorem (Lasiecka-Tuffaha)

When $\rho = 0$, the semigroup $e^{At}$ and the control operator $B$ generated from the PDE model (1) in $\mathcal{H}$ satisfy the following estimate

$$
\|e^{At}Bg\|_{\mathcal{H}_{-\delta}} \leq \frac{C_T}{t^{1/4+\epsilon}} \|g\|_{L_2(\Gamma_s)}
$$

(7)

for every $g \in L_2(\Gamma_s)$, and any $\delta > 0$, where

$$
\mathcal{H}_{-\delta} \equiv H \times H^{1-\delta}(\Omega_s) \times H^{-\delta}(\Omega_s), \quad \delta \geq 0
$$
Visco-Elastic Case 1: Without Dirichlet Control: $g_0 = 0$

- $A : E \rightarrow E'$

$$ (Au, \phi)_{\Omega_f} = -\langle \epsilon(u), \epsilon(\phi) \rangle_{\Omega_f}, \ \phi \in E $$

- Neumann map $N : L_2(\Gamma_s) \rightarrow H$

$$ Ng = h \iff \{ h \in H : \langle \epsilon(h), \epsilon(\phi) \rangle_{\Omega_f} = \langle g, \phi \rangle_{\Gamma_s}, \ \phi \in E \} $$

- 

$$ A = \begin{pmatrix}
A - L & AN\sigma() \cdot \nu & \rho AN\sigma() \cdot \nu \\
0 & 0 & I \\
0 & \text{div } \sigma() & \rho \text{div } \sigma()
\end{pmatrix} \quad (8) $$

$$ D(A) = \{(u, v, z) \in H : u \in E, \ A(u + N\sigma(v + \rho z) \cdot \nu) - Lu \in H, \ z \in H^1(\Omega_s), \ \text{div } \sigma(v + \rho z) \in L_2(\Omega_s), \ z|_{\Gamma_s} = u|_{\Gamma_s} \text{ in } H^{1/2}(\Gamma_s) \} $$

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Theorem 1.

For $\rho > 0$, the operator $A$ generates a strongly continuous analytic semigroup on $\mathcal{H}$.

Idea:

- There exist a constant $C > 0$ and $\omega > 0$, such that
  \[ |R(\lambda, A)|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|\lambda|}, \quad \text{Re}\lambda > \omega \]

- Change of Variable:
  \[ \zeta \equiv \nu + \rho \nu_t \]
New PDE system

\[
\begin{align*}
  u_t - \Delta u - L_u + \nabla p &= 0, \quad \text{div } u = 0, \\
  v_t + \rho^{-1} v &= \rho^{-1} \zeta, \\
  \zeta_t - \rho \text{div } \sigma(\zeta) &= \rho^{-1}(\zeta - v)
\end{align*}
\]

Boundary conditions

\[
\begin{align*}
  \sigma(\zeta) \cdot \nu &= (\epsilon(u) - p) \cdot \nu, \quad \text{on } \Sigma_s, \\
  \zeta &= v + \rho u, \quad \text{on } \Sigma_s, \quad u = 0 \quad \text{on } \Sigma_s
\end{align*}
\]
Visco-Elastic Case 2: With Dirichlet Control: $g_0 \neq 0$

- $A_D : E \to E'$:

\[(A_D u, \phi)_{\Omega_f} = -(\varepsilon(u), \varepsilon(\phi))_{\Omega_f}, \quad \phi \in E\]

\[\mathcal{D}(A_D) = \{u \in E : u|_{\Gamma_s} = 0\}\]

- Dirichlet Map $D : L_2(\Gamma_s) \to H^{1/2}(\Omega_s)$:

\[Dg = h \iff \{h \in H^{1/2}(\Omega_s) : \text{div} \sigma(h) = 0, \ h|_{\Gamma_s} = g\}\]

\[A_D = \begin{pmatrix} A - L_f & A\nu \sigma() & \rho A \nu \sigma() \\ 0 & 0 & 1 \\ -\rho A_D D & A_D - A_D D & \rho A_D \end{pmatrix} \quad (12)\]

\[\mathcal{D}(A_D) = \{(u, v, z) \in \mathcal{H} : u \in E, \ A(u + N\sigma(v + \rho z) \cdot \nu) - Lu \in \mathcal{H}, \ z \in H^1(\Omega_s), \text{div} \sigma(z) \in L_2(\Omega_s), \ z|_{\Gamma_s} = u|_{\Gamma_s} \text{ in } H^{1/2}(\Gamma_s)\}\]
\[ A_1 = \begin{pmatrix} A - L & 0 & AN\sigma() \cdot \nu \\ 0 & -\rho^{-1}I & \rho^{-1}I \\ 0 & -\rho^{-1}I & \rho \text{div } \sigma() + \rho^{-1}I \end{pmatrix} \]  

\text{Domain:} 

\[ \mathcal{D}(A_1) = \{(u, v, \zeta) \in \mathcal{H} : u \in E, \ A(u + N\sigma(\zeta) \cdot \nu) \in H, \zeta \in H^1(\Omega_s), \text{div } \sigma(\zeta) \in L_2(\Omega_s), \zeta|_{\Gamma_s} = [v + \rho u]|_{\Gamma_s} \text{ in } H^{1/2}(\Gamma_s)\} \]
Theorem 2.

The resolvent $R(\lambda, A)$ satisfies that $R(\lambda, A)B \rightarrow \mathcal{L}(U, \mathcal{H})$ for all $\lambda$ in the resolvent set of $A$. 
\[ B = \begin{pmatrix} AN \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix} \] (10)

**Theorem 3.**

For all \( \rho > 0 \), the PDE system (5) can be modeled abstractly as follows:

\[ y_t = Ay + Bg + Gw \in [D(A^*)]', \quad y_0 \in \mathcal{H} \] (11)

where \( A, B, G \) are defined as in (8) and (10). Moreover, \( A \) is the infinitesimal generators of s.c. analytic semigroup \( e^{At} \) on \( \mathcal{H} \) with the domain \( D(A) \) defined in (9).
Theorem 4.

If \( g_0 \neq 0 \), for all \( \rho > 0 \), the PDE system (1) can be modeled abstractly as follows:

\[
y_t = A_D y + B_D g + G_D w \in [D(A^*)]', \quad y_0 \in H
\]  

where \( A_D, B_D \) and \( G_D \) are defined in (12) and (14), \( g = (g_0, g_1) \), and \( A_D \) is the infinitesimal generators of s.c. analytic semigroup \( e^{A_D t} \) on \( H \) with the domain \( D(A_D) \) defined in (13).
Theorem 5.

The resolvent $R(\lambda, A_D)$ satisfies $R(\lambda, A_D)B_D \in \mathcal{L}(U \rightarrow \mathcal{H})$ for all $\lambda$ in the resolvent set of $A_D$. 
Main Theorem 1

1. In the viscoelastic case \((\rho > 0)\) with Neumann control only \((g_0 = 0)\), \(A, B\) and \(G\) satisfies the singular estimate, for \(g = (g_0, g_1)\).

\[
\|e^{At}Bg\|_\gamma \leq \frac{C_T}{t^{1/4+\theta+\epsilon}} \|g\|_{L^2(\Gamma_s)}, \quad 0 < t \leq T.
\] (16)

2. In the viscoelastic case \((\rho > 0)\) with both Dirichlet Control \((g_0 \neq 0)\) and Neumann control. The system also satisfies the singular estimate. Then for \(g = (g_0, g_1)\)

\[
\|e^{At}Bg\|_\gamma \leq \frac{C_T}{t^{3/4+\epsilon}} \|g\|_{H^{1/2}(\Gamma_s)}, \quad 0 < t \leq T.
\] (17)

Idea:

1. Finding the structure of the adjoint \(B^*\).

2. Use Theorem 1, \(A\) is analytic. Use interpolation and Theorem 2 and Theorem 5: \(R(\lambda, A)B \in \mathcal{L}(U, \mathcal{H})\) to estimate \(e^{At}B\).
Main Theorem 2

Let's assume that the static disturbance $w$ is in a bounded set in $L_2(\Omega_s)$. In reference to model (1) with $\rho > 0$ and the min-max game problem, there exists a critical $\gamma_c > 0$, for each initial condition in $\mathcal{H}$, that is,

$$y_0 = (u_0, v_0, v_1) \in \mathbb{H} \times H^1(\Omega_s) \times L_2(\Omega_s) = \mathcal{H},$$

- **Nonexistence.** if $0 < \gamma < \gamma_c$, $J(u, v, g, w) \to \infty$ as we take supremum over $w = (w_1, w_2, w_3) \in L_2(\mathcal{V})$. Thus there is not finite solution for (6) for any initial condition $y_0 \in \mathcal{H}$. 
Main Theorem 2 (Continue)

- **Existence.** If $\gamma > \gamma_c$, then for each initial condition $y_0 \in \mathcal{H}$ there exists a unique control $g^* = (g_0^*, g_1^*) \in C([0, T], H^{1/2}(\Gamma_s) \times L_2(\Gamma_s))$, a unique disturbance $(w_1^*, w_2^*, w_3^*) \in C([0, T], L_2(\Omega_f) \times L_2(\Omega_s) \times L_2(\Gamma_s))$ and the corresponding optimal state

$$y^*(t) = (u^*(t), v^*(t), v_t^*(t)) \in C([0, T], H \times H^1(\Omega_s) \times L_2(\Omega_s))$$

such that

$$J(y^*, g^*, w^*) = \sup_{w \in L_2(V)} \inf_{g \in L_2(U)} J(y(g, w), g, w)$$
Main Theorem 2 (Continue)

Furthermore, for all $t \in [0, T]$

- $\|g_1^*(t)\|_{L_2(\Gamma_s)} + \rho \|g_0^*(t)\|_{H^{1/2}(\Gamma_s)} \leq C|y_0|_\mathcal{H}$.
- $\|w_1^*(t)\|_{L_2(\Omega_r)} + \|w_2^*(t)\|_{L_2(\Omega_s)} + \|w_3^*(t)\|_{L_2(\Gamma_s)} \leq C|y_0|_\mathcal{H}$.
- $\|v^*(t)\|_{H^1(\Omega_s)}^2 + \|v_t^*(t)\|_{L^2(\Omega_s)}^2 + \|u^*(t)\|_{H^1}^2 \leq C|y_0|_\mathcal{H}$.

**Feedback Synthesis.** There exists a positive self-adjoint $n \times n$ operator matrix $P(t)$ on $\mathcal{H}$, let $P(t)(v^*(t), v_t^*(t), u^*(t)) = (p_1(t), p_2(t), p_3(t))$ with $p_1(t)$, $p_2(t)$ and $p_3(t)$ being $n$-dimensional vector function, such that the control is given by

$$g^*(t) = -B^*P(t)y^*(t) = -(p_1(t)|_{\Gamma_s}, \rho \Lambda^{-1}_\tau \sigma(p_3(t)) \cdot v)$$

$$w^*(t) = -\gamma^{-2}g^*P(t)y^*(t) = -\gamma^{-2}[p_1(t), p_3(t), p_1(t)|_{\Gamma_s}]$$

where $\Lambda_\tau$ is a positive first order tangential operator on $\Gamma_s$: $\|\Lambda_\tau^{1/2} u\|_{L_2(\Gamma_s)} = \|u\|_{H^{1/2}(\Gamma_s)}$. 
Main Theorem 2 (Continue)

- **Boundedness of the Gains.** The feedback operator $B^* P(t) \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ for all $0 \leq t \leq T$ with the estimate

$$
\|B^* P(t)y\|_\mathcal{U} \leq C \|y\|_\mathcal{H}, \quad \|G^* P(t)y\|_{L^2(\mathcal{V})} \leq C \|y\|_\mathcal{H}
$$

- **Riccati Equation.** $P(t)$ is the unique solution to the following non-standard Riccati equation: $P(T) = 0$ and

$$
\begin{aligned}
\left \{ \begin{array}{l}
\dot{P}(t)x, y\|_{\mathcal{H}} = -(R_1 x, R_2 y)_{\mathcal{H}} - (P(t)x, A y)_{\mathcal{H}} - (P(t) A x, y)_{\mathcal{H}} + (B^* P(t)x, B^* P(t)y)_{\mathcal{U}} - \gamma^{-2}(G^* P(t)x, G^* P(t)y)_{\mathcal{V}}
\end{array} \right.
\end{aligned}
$$

where $R = (R_1, R_2, R_3)$ with $R_1 \in \mathcal{L}(L^2(\Omega_f))$, $R_2 \in \mathcal{L}(H^1(\Omega_s))$, $R_3 \in \mathcal{L}(L^2(\Omega_s))$. 
Thank You