Geometry of the 3-body problem

- Joint w Rick Moeckel. U of Minn.
Lemaitre map = octahedral gem

\[ \pi'(\text{coll. pt}) = 2 \pi \text{ w/ ip pts.} \]

\[ \pi'(\text{gen. pt}) = 41 \text{ pts.} \]
Geometry of the 3-body problem

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FLOWS

Typical theory assumptions

Complete

Compact
phase space

Non-degenerate
periodic orbits

N-body Practice:

Incomplete (Singualrities: $r_{ij} \to 0$)

Non-Compact
phase space

Degenerate
periodic orbits (Symmetries)
Goal: for the N-body problem

REGULARIZE. Eliminate all collision singularities: regularize binary collisions, blow up triple or higher collisons, thereby COMPLETING the flow.

REDUCE. Quotient the phase space by the symmetry group. (Reduced) periodic orbits have chance to be NON-DEGENERATE

COMPACTIFY. Add boundaries at the ends corresponding to escape.

REGULARIZATION & REDUCTION done for the planar 3 body problem.
Partial progress: spatial 3 and planar 4 body problem

COMPACTIFICATION: open; prelim. work by C. Robinson
Partial History: \[ (d, N) = (\text{dim. of space}, \text{Number of bodies}) \]

[1772]. Lagrange. Reduced: \( d \) arbitrary, \( N = 3 \).

**![1921]. Levi-Civita. Regularizes single binary collisions in the plane \((d=2)\). Cx var. methods. Good for perturbed Kepler.

**![1954]. Lemaitre. Regularizes & reduces planar three-body problem \((d,N) = (2,3)\); (origins of formulae, geometry unclear.)

[1965]. Kuustanheimo-Steifel. Extends L-Cs [1921] reg. binary collisions in space \((d=3)\) using quaternions (`spinors').

**![1970]. Heggie. Democratic regularization. \( d = 2 \) or 3. \( N \) arbitrary.


(A): Levi-Civita Regularization. $d = 2$

$$H_{\kappa} = \frac{1}{2}|q|^2 - \frac{1}{|q|} = -\frac{1}{2}\alpha$$

$q \in \mathbb{C} = \mathbb{R}^2$

$$z^2 = q$$

2: 1 branched cover

$$\frac{d}{d\tau} = |q| \frac{d}{dt}; \quad r = |q|$$

Slow time down as approach collision $q=0$

Kepler:

$$\ddot{q} = -\frac{q}{|q|^3}$$

$$\Rightarrow \frac{d^2z}{d\tau^2} = H_{\kappa} z$$

harm. oscillator!

Cut open and unfold
Heggie: $Q_{ij} = q_i - q_j = -Q_{ji}$ \text{(translation-invariant cx coordinates)}

triangle constraints: $Q_{ij} + Q_{jk} + Q_{ki} = 0$

$\implies$ Cx linear space of dim. $N - 1$

cp. reduction by translations

with binary collision of $i$ and $j$: $Q_{ij} = 0$

Apply L-C to each

$$z_{ij}^2 = Q_{ij} ; \quad \frac{d}{d\tau} = \prod_{i < j} r_{ij} \frac{d}{dt} ; \quad r_{ij} = |Q_{ij}|$$

Democratic regularization of all binary collisions regularized!

Next: reduce by rotations.
Easier: Reduce by Rotations AND Scalings:

\[ q_j \mapsto \lambda q_j; \quad Q_{jk} \mapsto \lambda Q_{jk}; \quad \lambda = \rho e^{i\theta}, \quad \rho > 0 \]

scaling \quad rotation

\[ z_{ik}^2 + z_{jk}^2 + z_{ki}^2 = 0 \]

\[ \lambda = \mu^2; \quad z_{jk} \mapsto \mu z_{jk} \]

\[ z_{jk} \mapsto [z_{jk}] \]

\[ Q_{jk} = z_{jk}^2 \mapsto [Q_{jk}] \]

\[ Q^{N-1} \subset \mathbb{C}^{(N \choose 2)} \rightarrow \mathcal{V} \subset \mathbb{CP}^{D(N)} \]

\[ \mathbb{C}^{N-1} \subset \mathbb{C}^{(N \choose 2)} \rightarrow \mathbb{CP}^{N-2} \subset \mathbb{CP}^{D(N)} \]

generalized Lemaitre map has degree \(2^{D(N)}\) where: \[ D(N) = \left(\frac{N}{2}\right) - 1 \]
Put scale back in: \[ R = \sqrt{I}; I = \sum m_i m_j r_{ij}^2 / \sum m_i = \sum m_i |q_i|^2 \]

\[ \Rightarrow \text{2nd order ODEs in } (R, [Q_{ij}]) \text{ or } (R, [z_{ij}]) \]
(2,3) CASE

\[ z_{ij}^2 = Q_{ij} \]

\[ \frac{d}{d\tau} = r_{12}r_{23}r_{31} \frac{d}{dt} \]

\[ Q_{12} + Q_{23} - Q_{31} = 0 \]

\[ \Rightarrow z_{12}^2 + z_{23}^2 + z_{31}^2 = 0 \]

Regularized shape sphere = \{ \[ z_{12}, z_{23}, z_{31} \] \in \mathbb{CP}^2 : z_{12}^2 + z_{23}^2 + z_{31}^2 = 0 \}

\[ Q_{ij} = z_{ij}^2 \]

\[ Q_{ij} \rightarrow [z_{ij}] \]

\[ \begin{align*}
\text{Cone in complex 3-space} & \quad \Rightarrow \quad \text{(*) conic in proj. plane = regularized shape sphere} \\
\text{Complex 2-space} & \quad \Rightarrow \quad \{ [Q_{12}, Q_{23}, Q_{31}] \in \mathbb{CP}^2 : Q_{12} + Q_{23} + Q_{31} = 0 \}
\end{align*} \]

\[ (\star) \text{ parameterizing the conic (a la Pythagorean triples)} \]

\[ z_{12} = 2i \ x_1 x_2 \quad z_{31} = x_1^2 + x_2^2 \quad z_{23} = i(x_1^2 - x_2^2) \]
Lemaitre map = octahedral gem

\[ \pi' (\text{coll. pt}) = \text{two indip pts.} \]

\[ \pi'/\text{gen. pt)} = 41 \text{ pts.} \]
Some orbits plotted in reduced space

Figure eight orbit of Chenciner and Montgomery

(contours of $\sqrt{IU}$)
Some orbits plotted in reduced space

Figure eight orbit of Chenciner and Montgomery

(contours of \(\sqrt{IU}\))
Some orbits plotted in reduced space

Figure eight orbit of Chenciner and Montgomery

(contours of $\sqrt{IU}$)
Some Three-Body Orbits in the Regularized Configuration Space

Figure-eight orbit

The orbit in regularized shape space is remarkably simple!