Mathematical Modeling and Simulation of Pedestrian Motion

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Pedestrian motion

- Empirical study of human crowds started about 50 years ago.
- Nowadays there is a large literature on different micro- and macroscopic approaches available.
- Challenges: microscopic interactions not clearly defined, multiscale effects, finite size effects,.....
Mathematical modeling - microscopic level

1. **Force based models**: position of a particle is determined by forces acting on it e.g. Newton dynamics, stochastic differential equations (SDE), ....
   Newton equations of motion
   \[
   \frac{dX_i}{dt} = V_i \\
   \frac{dV_i}{dt} = F_i(X_1, \ldots, X_N, V_1, \ldots, V_N) + \sigma_i dB_i.
   \]
   \(X_i = X_i(t)\) location of the i-th particle, \(V_i = V_i(t)\) it its velocity, \(F_i\) forces acting on it.

2. **Stochastic optimal control**
   \[
   \frac{dX_i}{dt} = v_i dt + \sigma_i dB_i.
   \]
   Each agent wants to minimize a cost functional
   \[
   J_i(v_1, v_2, \ldots, v_N) = \mathbb{E}( \int_0^T L_i(X_i, v_i) + F(X_1, \ldots, X_N) dt)
   \]
   where \(L\) and \(F\) denote the running cost.

3. **Lattice based models**: No time today ....
Nonlinear diffusion transportation models

- Macroscopic limit $N \to \infty$ one usually obtains a nonlinear transport-diffusion equations of the form

$$\rho_t = \text{div}(D(\rho) \nabla (E'(\rho) - V + W \ast \rho)).$$

- $V = V(x)$ is an external potential energy (e.g., confinement, ...),
- $D = D(\rho)$ denotes the nonlinear diffusion/mobility
- $E = E(\rho)$ an entropy/internal energy.
- $W = W(x)$ is an interaction energy.

- Classic fluid dynamic based models: Based on the conservation of mass

$$\rho_t = \text{div}(\rho v)$$

with a particularly chosen velocity $v$ (which depends on the specific modeling assumptions).
Examples: Traffic flow (e.g., Lighthill-Whitman and Richards model), Hughes model for pedestrian flow ....
Hughes model for pedestrian flow

1. Speed of pedestrians depends on the density of the surrounding pedestrian flow
   \[ v = f(\rho)u, \quad |u| = 1. \]

2. Pedestrians have a common sense of the task (called potential \( \phi \))
   \[ u = -\frac{\nabla \phi}{|\nabla \phi|}. \]

3. Pedestrians try to minimize their travel time, but want to avoid high densities
   \[ |\nabla \phi| = \frac{1}{g(\rho)f(\rho)}. \]

Hughes model for pedestrian flow for \( g(\rho) = 1 \) and \( f(\rho) = \rho - \rho_{\text{max}} \)

\[ \rho_t - \text{div}(\rho(\rho - \rho_{\text{max}})^2 \nabla \phi) = 0, \]
\[ |\nabla \phi| = \frac{1}{\rho - \rho_{\text{max}}} \]

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Mean field games

Microscopic model

- N-player stochastic differential game

\[ \inf_{v_i \in A} \mathbb{E} \left[ \int_0^T f(t, X_i, v_i, \rho) \, dt + g(\rho, X_i, t = T) \right] \]
\[ dX_i = v_i \, dt + \sigma \, dB_i, \quad X_i(t = 0) = x. \]

- Transient macroscopic model

Calculate Nash equilibrium, limiting equations as \( N \to \infty \) gives time dependent mean field game: Find \((\phi, \rho)\) such that

\[ \frac{\partial \phi}{\partial t} + \nu \Delta \phi - H(x, \nabla \phi) = V[\rho] \]
\[ \frac{\partial \rho}{\partial t} - \nu \Delta \rho - \text{div} \left( \frac{\partial H}{\partial \rho}(x, \nabla \phi) \rho \right) = 0, \]

with the initial and end conditions

\[ \phi(x, T) = g[\rho(x, T)], \quad \rho(x, 0) = \rho_0(x), \]

where \( H \) is the Legendre transform of the running cost \( f \).

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Link to optimal control problems

If the running cost $f$ has the form

$$f(x, t, v, \rho) = L(x, t, v)\rho + \Phi[x, \rho],$$

and $V = \Phi'$, then the MFG can be written as an optimal control problem. For example let $f(x, t, v) = \frac{1}{2}\rho|v|^2 + \Phi(x, \rho)$, then

$$\inf_v \left[ \int_0^T \int_{\Omega} \frac{1}{2}\rho|v|^2 + \Phi(x, \rho) \, dxdt + g(\rho(T)) \right]$$

under the constraint that

$$\frac{\partial \rho}{\partial t} = \nu \Delta \rho - \text{div}(\rho v),$$

$$\rho(x, 0) = \rho_0(x).$$

The formal optimality condition is $v = \nabla \phi$ and therefore the adjoint equation reads as

$$\frac{\partial \phi}{\partial t} + \nu \Delta \phi - \frac{1}{2}|\phi|^2 = V(\rho)$$

with the terminal condition $\phi(x, T) = g'(\rho(T))$. 
An optimal control approach for fast exit scenarios

- Let us consider an evacuation or fast exit scenario, i.e. a room with one or several exits from which a group wants to leave as fast as possible.
- Each individual tries to find the optimal trajectory to the exit, taking into account the distance to the exit, the density of people and other costs.

Figure: Fast-exit experiment conducted at the TU Delft
Fast Exit of Particles

- We consider a deterministic particle (of unit mass), which wants to leave a domain $\Omega$ as fast as possible. Let $X(t)$ denote the particle trajectory and

$$T_{exit}(X) = \sup\{t > 0 \mid X(t) \in \Omega\}.$$

- Fastest path is chosen such that the following weighted functional of the exit time and the kinetic energy

$$\frac{1}{2} \int_0^{T_{exit}} |V(t)|^2 \, dt + \frac{\alpha}{2} T_{exit}(X),$$

subject to $\dot{X}(t) = V(t), \ X(0) = X_0$, is minimized over $(X, V)$.

- If we consider a stochastic particle and write $d\mu = \rho \, dx$ the minimization reads as

$$l_T(\rho, v) = \frac{1}{2} \int_0^T \int_\Omega \rho(x, t) |v(x, t)|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega \rho(x, t) \, dx \, dt,$$

subject to $\partial_t \rho + \nabla \cdot (\nu \rho) = \frac{\sigma^2}{2} \Delta \rho, \ \rho(x, 0) = \rho_0(x)$. 
Optimality conditions

- The optimality conditions can be (formally) calculated via the Lagrangian with dual variable $\phi$, i.e.

$$L_T(\rho, v, \phi) = l_T(\rho, v) + \int_0^T \int_{\Omega} \left( \partial_t \rho + \nabla \cdot (v \rho) - \frac{\sigma^2}{2} \Delta \rho \right) \phi \, dx \, dt.$$

- For the optimal solution we have

$$0 = \partial_v L_T(\rho, v, \phi) = \rho v - \rho \nabla \phi$$

$$0 = \partial_\rho L_T(\rho, v, \phi) = \frac{1}{2} |v|^2 + \frac{\alpha}{2} - \partial_t \phi - v \cdot \nabla \phi - \frac{\sigma^2}{2} \Delta \phi,$$

with the additional terminal condition $\phi = 0$ at $t = T$.

- Inserting $v = \nabla \phi$ we obtain the following system, which has a similar structure as a MFG:

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) - \frac{\sigma^2}{2} \Delta \rho = 0$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \frac{\sigma^2}{2} \Delta \phi = \frac{\alpha}{2}.$$
Mean field games and crowding

- We consider the following generalization of the optimal control problem

\[ I_T(\rho, v) = \frac{1}{2} \int_0^T \int_\Omega F(\rho)|v(x, t)|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega E(\rho) \, dx \, dt, \]

subject to

\[ \partial_t \rho + \nabla \cdot (G(\rho)v) = \frac{\sigma^2}{2} \Delta \rho, \text{ with initial condition } \rho(x, t = 0) = \rho_0(x). \]

Motivation for this generalization

- The function \( G = G(\rho) \) corresponds to a nonlinear mobility, e.g. \( G(\rho) = \rho(\rho_{\text{max}} - \rho) \). Nonlinear mobilities have been derived and used in different settings and applications of crowded motion, e.g. ion channels or cell biology.

- The function \( F = F(\rho) \) corresponds to transport costs created by large densities. Particular choice may be

\[ F(\rho) \to \infty \text{ as } \rho \to \rho_{\text{max}}. \]

- A nonlinear function \( E = E(\rho) \) may model active avoidance of jams in the exit strategy, in particular by penalizing large density regions.
Relation to the classical model by Hughes

- We choose $H(\rho) = \frac{G^2}{F} = \rho f(\rho)^2$ and $E(\rho) = \alpha \rho$ to obtain the following optimality conditions:

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla \phi) = 0$$

$$\partial_t \phi + \frac{f(\rho)}{2} (f(\rho) + 2 \rho f'(\rho)) |\nabla \phi|^2 = \frac{\alpha}{2}$$

- For large $T$ we expect equilibration of $\phi$ backward in time. If $\sigma = 0$, then for time $t$ of order one the limiting model becomes

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla \phi) = 0$$

$$(f(\rho) + 2 \rho f'(\rho)) |\nabla \phi|^2 = \frac{c}{f(\rho)},$$

which is almost the Hughes model.

- For $f(\rho) = \rho_{\text{max}} - \rho$ we have

$$f(\rho) + 2 \rho f'(\rho) = \rho_{\text{max}} - 3 \rho,$$

thus for small densities the behavior is similar, but the singular point is $\rho = \frac{\rho_{\text{max}}}{3}$. 
Boundary conditions

- On the Neumann boundary $\Gamma_N$ we clearly have no outflux, hence naturally

$$\left(-\frac{\sigma^2}{2}\nabla \rho + j\right) \cdot n = 0.$$

- At an exit $\Gamma_E$: the outflux depends on how fast people can leave the room. Let $\beta$ denote the rate of passing, then the outflow is proportional to $\beta \rho$, i.e.

$$\left(-\frac{\sigma^2}{2}\nabla \rho + j\right) \cdot n = \beta \rho.$$

- Boundary conditions for the adjoint variable $\phi$ can be obtained via Lagrangian and result in

$$\frac{\sigma^2}{2} \nabla \phi \cdot n + \beta \phi = 0 \text{ on } \Gamma_E \quad \text{and} \quad -\frac{\sigma^2}{2} \nabla \phi \cdot n = 0 \text{ on } \Gamma_N.$$
Analysis of the optimal control model

We set \( F = G = H \) and make the following assumptions on \( F \) and \( E \):

(A1) \( F = F(\rho) \in C^1(\mathbb{R}), \) \( F \) bounded, \( E = E(\rho) \in C^1(\mathbb{R}) \) and \( F(\rho) \geq 0, E(\rho) \geq 0 \) for \( \rho \in \mathbb{R} \).

Existence of minimizers is guaranteed if

(A2) \( E = E(\rho) \) is convex.

We fix the maximal density \( \rho_{\max} > 0 \) and denote by \( \mathcal{Y} = [0, \rho_{\max}] \). To force the minimizers of the problem to satisfy \( \rho \in \mathcal{Y} \), we extend \( F \) to \( F(\rho) = 0 \) on \( \rho \in \mathbb{R} \setminus \mathcal{Y} \). This defines the next assumption on \( F \), i.e.

(A3) \( F(0) > 0 \) if \( \rho \in \mathcal{Y} \) and \( F = 0 \) otherwise.

Uniqueness holds for:

(A4) \( F = F(\rho) \) is concave.

We consider the optimization problem on the set \( V \times Q \), i.e. \( f_T(\rho, v) : V \times Q \to \mathbb{R} \), where \( V \) and \( Q \) are defined as follows:

\[
V = L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \quad \text{and} \quad Q = L^2(\Omega \times (0, T)).
\]
Analysis of the optimal control problem

- Then the optimization problem reads as

\[
\min_{(\rho, v) \in V \times Q} I_T(\rho, v) \quad \text{such that} \quad \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \text{div}(F(\rho)v),
\]

respectively in momentum formulation

\[
\min_{(\rho, j) \in V \times Q} I_T(\rho, j) \quad \text{such that} \quad \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \text{div}(j).
\]

- In the case of nonconcave $H$ we use a different formulation based on the variable $w = \sqrt{F(\rho)}v$. Then the minimization problem becomes:

\[
\min_{(\rho, w) \in V \times Q} \frac{1}{2} \int_0^T \int_{\Omega} (|w|^2 + E(\rho)) \, dx \, dt \quad \text{such that} \quad \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \text{div}(\sqrt{F(\rho)}w).
\]
Existence results

Theorem (Existence in the general case)

Let $\rho_0 \in L^2(\Omega)$. Let (A1) and (A2) be satisfied and let $\sigma > 0$. Then the variational problem has at least a weak solution $(\rho, w) \in V \times Q$ with initial condition $\rho_0$. If in addition (A3) is satisfied, then $\rho \in \mathcal{T}$.

Theorem (Existence for concave mobility)

Let (A1), (A2), (A3), and (A4) be satisfied and let $\sigma > 0$. Then the variational problem has at least one minimizer $(\rho, j) \in L^\infty(\Omega \times (0, T)) \times Q$ such that $\rho(x) \in \mathcal{T}$ for almost every $x \in \Omega$. If $E$ is strictly convex the minimizer is unique.
We solve the optimal control problem using a steepest descent method given by

1. Solve the forward equation for $\rho = \rho(x, t)$

\[
\frac{\partial_t \rho}{2} = \Delta \rho - \nabla \cdot (F(\rho) \nu)
\]

\[
\left(-\frac{\sigma^2}{2} \Delta \rho + \nabla \cdot F(\rho) \nu\right) \cdot n = \beta \rho \text{ on } \Gamma_E \text{ and } \left(-\frac{\sigma^2}{2} \Delta \rho + \nabla \cdot F(\rho) \nu\right) \cdot n = 0 \text{ on } \Gamma_N,
\]

using Newton’s method for the implicit Euler discretization and a mixed hybrid DG method for the discretization in space.

2. Calculate the backward evolution of the adjoint variable $\phi$ using the density $\rho$

\[
- \partial_t \phi - \frac{\sigma^2}{2} \Delta \phi - G'(\rho) \nu \cdot \nabla \phi = -\frac{1}{2} F'(\rho) |\nu|^2 - \frac{1}{2} E'(\rho)
\]

\[
- \frac{\sigma^2}{2} \nabla \phi \cdot n - \beta \phi = 0 \text{ on } \Gamma_E \text{ and } - \frac{\sigma^2}{2} \nabla \phi \cdot n = 0 \text{ on } \Gamma_N,
\]

using an implicit in time discretization and a mixed hybrid DG method.

3. Update the velocity $\nu = \nu - \tau (F(\rho) \nu - G(\rho) \nabla \phi)$.

4. Go to (1) until convergence of the functional.
Fast exit for three groups

(a) Solution of the classical Hughes model

(b) Solution of the mean field optimal control approach
Future work

• Analysis of the problem for more general functions $E$, $F$ and $G \Rightarrow$ non convex optimization problems
• Simulations in 2D, numerical solvers for non-convex optimization problems.
• Other generalization of the Hughes model for pedestrian flow, e.g. include local vision.
• Multiscale problems in crowd dynamics: crowd-leader interactions or the motion of small social groups in large crowds.
• Inverse problems: How many leaders are necessary to guide a crowd efficiently?

Thank you very much for your attention!