Mathematical modeling of the HIV/AIDS epidemic in Cuba

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AIDS was first reported on June 5, 1981 by the CDC.
Highest prevalence is in sub-Saharan Africa (5%).
Caribbean region has second highest prevalence.
As of 2010, 60 mil HIV infected, 30 mil AIDS deaths.
In 2011, there were 34 mil people living with HIV.
Newly infected: 3.2 mil in 2001, 2.5 mil in 2011.
AIDS deaths: Peak of 2.3 mil in 2005, 1.7 mil in 2011.
What is AIDS?

An HIV-infected individual has AIDS if

- He/She has fewer than 200 T-lymphocytes per microliter OR
- One or more of 26 various diseases including
  - Kaposi’s sarcoma, lymphoma, candidiasis, etc.

Symptoms: fever, weight loss, night sweats, diarrhea.
HIV/AIDS in Cuba

- HIV prevalence is 0.2%.
- 99% of transmissions are through sexual relations.
- 77-80% of HIV infected are men.
- Average of 1.6 mil tests performed each year.
- Antiretroviral therapy (ARV) coverage is 100%.
- In 1983 Cuba initiated program to control HIV/AIDS.
National AIDS Commission

1. Design a national HIV prevention program
2. Develop efforts for prevention of vertical transmission
3. Undertake epidemiological surveillance and control
4. Spearhead scientific research and development
5. Establish a national sanatorium network

“Health is a human right.”
HIV/AIDS data for Cuba

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<th>Year</th>
<th>HIV cases</th>
<th>AIDS cases</th>
<th>Death due to AIDS</th>
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Compartments

1. $S(t)$: the susceptible population
2. $X(t)$: undiagnosed HIV infected people
3. $Y(t)$: diagnosed HIV infected people
4. $Z(t)$: people diagnosed with AIDS

Earlier mathematical models:
- de Arazoza and Lounes (2002)
- Rapatski et al. (2006)
Parameters and model

1. $\lambda$: recruitment rate of the susceptible class.
2. $\alpha$: transmission rate of HIV+ by sexual transmission with $X$.
3. $\hat{\beta}$: rate at which HIV-infected class develop AIDS.
4. $k$: rate at which $X$ class are diagnosed through contact tracing.
5. $\hat{k}$: rate at which $X$ are diagnosed through random testing.
6. $\mu$: mortality rate of the adult class.
7. $\hat{\mu}$: mortality rate of the population with AIDS.

Model equations:

\[
\begin{align*}
\dot{S} &= \lambda - \alpha X S - \mu S \\
\dot{X} &= \alpha X S - kXY - (\mu + \hat{\beta} + \hat{k}) X \\
\dot{Y} &= kXY + \hat{k}X - (\mu + \hat{\beta}) Y \\
\dot{Z} &= \hat{\beta} (X + Y) - \hat{\mu} Z
\end{align*}
\]
Basic reproduction number $R_0$ is the number of secondary infections caused by an infectious individual that enter a fully susceptible population. $R_0$ is determined by computing the spectral radius of the matrix formed by the product of the next generation matrix, $F$, and the inverse of the transition matrix, $V$, given by

$$F = \begin{pmatrix} \alpha \frac{\lambda}{\mu} & 0 & 0 \\ k & 0 & 0 \\ \hat{\beta} & \hat{\beta} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\mu + \hat{\beta} + \hat{k}) & 0 & 0 \\ 0 & (\mu + \hat{\beta}) & 0 \\ 0 & 0 & \hat{\mu} \end{pmatrix}.$$ 

A routine computation yields

$$R_0 = \frac{\lambda \alpha}{\mu (\mu + \hat{\beta} + \hat{k})}.$$
Disease-free equilibrium

The model has a disease-free equilibrium (DFE), $E_0 = (\frac{\lambda}{\mu}, 0, 0, 0)$.

**Proposition**

1. $E_0$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.
2. $E_0$ is global asymptotically stable if $R_0 \leq 1$.

**Proof.**

$$J(E_0) = \begin{pmatrix}
-\mu & -\alpha \lambda / \mu & 0 & 0 \\
0 & \alpha \lambda / \mu - (\mu + \hat{k} + \hat{\beta}) & 0 & 0 \\
0 & \hat{k} & -\mu - \hat{\beta} & 0 \\
0 & \hat{\beta} & \hat{\beta} & -\mu
\end{pmatrix}$$

Eigenvalues are $h_1 = -\mu$, $h_2 = -\hat{\mu}$, $h_3 = - (\mu + \beta_2)$, and $h_4 = \alpha \lambda / \mu - (\mu + \hat{k} + \beta_1) = (R_0 - 1)(\mu + \hat{k} + \beta_1)$. 
Endemic equilibrium

The model has endemic equilibrium $E = (S^*, X^*, Y^*, Z^*)$ where

$$X^* = \frac{(\mu + \hat{\beta})Y^*}{\hat{k} + kY^*}, \quad S^* = \frac{\lambda}{\alpha X^* + \mu}, \quad Z^* = \frac{(X^* + Y^*)\hat{\beta}}{\hat{\mu}},$$

$Y^*$ is the positive root of

$$aY^2 + bY + c = 0,$$

(0.1)

and

$$a = k(\mu k + \alpha(\mu + \hat{\beta})) > 0$$

$$b = \alpha(\mu + \hat{\beta})(\mu + \hat{\beta} + \hat{k}) + k\mu\hat{k} + \frac{\lambda\alpha k}{R_0} - \lambda\alpha k$$

$$c = \hat{k}(\mu(\mu + \hat{\beta} + \hat{k}) - \lambda\alpha) = \hat{k}\mu(\mu + \hat{\beta} + \hat{k})(1 - R_0).$$
Endemic equilibrium, cont’d

Theorem

The system can have at most one positive equilibrium. More precisely,

1. If $R_0 > 1$, there exists a unique positive stable equilibrium $E = (S^*, X^*, Y^*, Z^*)$.
2. If $R_0 < 1$, there is no positive equilibrium.

Proposition

1. $E$ is locally asymptotically stable if $R_0 > 1$.
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Future work

\[
\begin{align*}
\dot{S} &= \lambda - [(\alpha_1 + \beta_1)X_1 + (\alpha_2 + \beta_2)X_2 + (\sigma_1 + \sigma_2)Y + (\omega_1 + \omega_2)Z - \mu] S \\
\dot{X}_1 &= (\alpha X_1 + \beta_1 X_2 + \sigma_1 Y + \omega_1 Z) S - kX_1 Y - (\mu + \bar{\beta} + \bar{k}) X_1 \\
\dot{X}_2 &= (\alpha_2 X_1 + \beta_2 X_2 + \sigma_2 Y + \omega_2 Z) S - (\mu + \bar{\beta} + \bar{k}) X_2 \\
\dot{Y} &= kX_1 Y + \hat{k}(X_1 + X_2) - (\mu + \bar{\beta}) Y \\
\dot{Z} &= \bar{\beta}(X_1 + X_2 + Y) - \hat{\mu} Z
\end{align*}
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- $X_1$: undiagnosed HIV+ infected via sexual transmission
- $X_2$: undiagnosed HIV+ infected via nonsexual transmission
- $\alpha_1, \ldots$: transmission rates of HIV+ by sexual transmission
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\dot{Z} &= \hat{\beta} (X_1 + X_2 + Y) - \hat{\mu} Z
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(0.2)

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Questions??