Dynamics of near parallel vortex filaments

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Outline

Vortex filaments

Natural questions in Hamiltonian dynamics

Hamiltonian PDEs

A variational formulation for invariant tori
Vortex filaments in $\mathbb{R}^3$

- one vortex filament:
  stationary linear filament, with uniform vortex strength $\gamma = 1$

$$b(s) = (0, 0, s)$$

It generates a flow field in $\mathbb{R}^3$ described by

$$u = (\partial_{x_2} \psi, -\partial_{x_1} \psi, 0)$$

which is given by a stream function

$$\psi = \frac{1}{2} \log(x_1^2 + x_2^2) = \frac{1}{2} \log(|z|^2)$$

where $z = x_1 + ix_2$ are complex horizontal coordinates.
Vortex filament pairs

Two exactly parallel linear vortex filaments evolve as described by point vortices in $\mathbb{R}^2$

- Opposite vorticity $\gamma_1 = 1 = -\gamma_2$, initial configuration
  
  $$ b_1(s) = \left( \frac{1}{2}a + i0, s \right), \quad b_2(s) = \left( -\frac{1}{2}a + i0, s \right) $$

  then ballistic linear evolution
  
  $$ b_1(s, t) = \left( \frac{1}{2}a + i\frac{t}{a}, s \right), \quad b_2(s, t) = \left( -\frac{1}{2}a + i\frac{t}{a}, s \right) $$

- Same vorticity $\gamma_1 = 1 = \gamma_2$ with the above initial configuration have circular orbits with angular frequency $\omega = a^{-2}$
  
  $$ b_1(s, t) = \left( \frac{1}{2}ae^{\frac{i\omega t}{a^2}}, s \right), \quad b_2(s, t) = \left( \frac{1}{2}ae^{\frac{i\omega t}{a^2} + \pi}, s \right) $$
**Question:** Consider two near-vertical vortex filaments, slightly perturbed from exactly vertical. Do there persist similar orbital motions, whose time evolution is periodic or quasi-periodic. Configuration to be $2\pi$ periodic in the vertical $x_3$ variables.

In ‘center of vorticity’ coordinates, the horizontal separation of the two vortex filaments is

$$w(s, t) = x_1(s, t) + ix_2(s, t)$$

Klein, Majda & Damodaran model of near parallel vortex filaments, in a frame rotating with angular frequency $\omega$

$$i \partial_t w + \partial_s^2 w - \omega w + \frac{w}{|w|^2} = 0$$  \hspace{1cm} (1)

**NB:** For configurations which are greatly deformed from vertical, this is not a good accurate model.
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$$i\partial_tw + \partial_s^2w - \omega w + \frac{w}{|w|^2} = 0 \quad (1)$$

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$$i\partial_t w + \partial_s^2 w - \omega w + \frac{w}{|w|^2} = 0 \quad (1)$$

NB: For configurations which are greatly deformed from vertical, this is not a good accurate model.
This is a PDE in Hamiltonian form. Set \( w = a + v(s, t) \) with \( a \in \mathbb{R} \) and \( v(s, t) \) a perturbation term,

\[
i \partial_t v + \partial_s^2 v - \omega(a + v) + \frac{a + v}{|a + v|^2} = 0
\]

(2)

by the choice \( \omega = a^{-2} \) then \( v = 0 \) is stationary.

The Hamiltonian is

\[
H = \int_0^{2\pi} \frac{1}{2} |\partial_s v|^2 + \frac{1}{2a^2} |a + v|^2 - \frac{1}{2} \log |a + v|^2 \, ds
\]

(3)

Writing \( v(s, t) = X(s, t) + iY(s, t) \) the dynamics are given by Hamilton’s canonical equations

\[
\partial_t X = \text{grad}_Y H
\]

\[
\partial_t Y = -\text{grad}_X H
\]

Small \( ||v||_{H^1} \) solutions exist globally in time (C. Kenig, G. Ponce & L. Vega (2003), V. Banica & E. Miot (2012))
Linearized equations

- The tangent plane approximation is given by linearization.

The linearized equations at equilibrium \((X, Y) = 0\) are derived from the quadratic Hamiltonian

\[
H^{(2)} = \int_0^{2\pi} \left[ \frac{1}{2} \left( \partial_s X \right)^2 + \left( \partial_s Y \right)^2 + \frac{2}{a^2} X^2 \right] ds
\]

(4)

- Linearized equations

\[
\partial_t X = \nabla_Y H^{(2)} = -\partial_s^2 Y
\]

\[
\partial_t Y = -\nabla_X H^{(2)} = \partial_s^2 X - \frac{2}{a^2} X
\]
Linear flow

- Writing in a Fourier basis and using the Plancherel identity
  \[ X(s) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \hat{X}_k e^{iks} \]
  \[ Y(s) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \hat{Y}_k e^{iks} \]

  \[ H^{(2)} = \sum_{k \in \mathbb{Z}} \frac{1}{2} \left( \left( k^2 + \frac{2}{a^2} \right) |\hat{X}_k|^2 + k^2 |\hat{Y}_k|^2 \right) \]

An infinite series of uncoupled harmonic oscillators, with frequencies \( \omega_k(a) = \pm k \sqrt{k^2 + \left( \frac{2}{a^2} \right)} \).

- The solution operator, or the linear flow

  \[
  \begin{pmatrix} X(s, t) \\ Y(s, t) \end{pmatrix} = \sum_{k \in \mathbb{Z}} e^{iks} \begin{pmatrix} \cos(\omega_k t) & k^2 \sin(\omega_k t)/\omega_k \\ -\omega_k \sin(\omega_k t)/k^2 & \cos(\omega_k t) \end{pmatrix} \begin{pmatrix} \hat{X}_k \\ \hat{Y}_k \end{pmatrix}
  \]

  \[
  := \Phi(t) \begin{pmatrix} X(s, 0) \\ Y(s, 0) \end{pmatrix}
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Angles evolve with linear motion \( \theta_k \mapsto \theta_k + n \omega_k \).
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  Angles evolve with linear motion \( \theta_k \mapsto \theta_k + t\omega_k \)
Elementary facts

1. All solutions are **Periodic**, or **Quasi-Periodic**, or in general **Almost Periodic** functions of time

2. More specifically, for initial data \((X^0, Y^0)\) the active wavenumbers are \(K := \{k : (\dot{X}^0_k, \dot{Y}^0_k) \neq 0\}\)

The dimension of the frequency basis is

\[
m := \dim_{\mathbb{Q}}(\text{span}_{\mathbb{Q}}\{\omega_k : k \in K\})
\]

3. Orbit space consists of tori

\[
\overline{\text{orbit}}(X^0, Y^0) = \overline{\{\Phi(t)(X^0, Y^0) : t \in \mathbb{R}\}} = \mathbb{T}^m
\]

**Periodic (P):** \(m = 1\)

**Quasi-Periodic (QP):** \(1 < m < +\infty\)

**Almost Periodic (AP):** \(m = +\infty\)

**NB:** For generic \(a\) then \(\omega_k(a)\) satisfy \(1 \leq m \leq +\infty\)
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  An infinite series of uncoupled harmonic oscillators, with frequencies \( \omega_k(a) = \pm k \sqrt{k^2 + \frac{2}{a^2}} \).

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Elementary facts

4. Energy is conserved

\[ H^{(2)}(X, Y) = H^{(2)}(\Phi(t)(X, Y)) \]

5. Indeed each action variable is conserved

\[ I_k = \frac{\sqrt{k^2 + (2/a^2)}}{2|k|} |X_k|^2 + \frac{|k|}{2\sqrt{k^2 + (2/a^2)}} |Y_k|^2 \]

because for each \( k \)

\[ \frac{d}{dt} \left( \begin{pmatrix} \hat{X}_k \\ \hat{Y}_k \end{pmatrix} \Phi_k(t)^T \begin{pmatrix} k^2 + \frac{2}{a^2} & 0 \\ 0 & k^2 \end{pmatrix} \Phi_k(t) \begin{pmatrix} \hat{X}_k \\ \hat{Y}_k \end{pmatrix} \right) = 0 \]

Hence all Sobolev energy norms are preserved

\[ H^{(2)} = \sum_{k \in \mathbb{Z}} \omega_k I_k, \quad \|(X, Y)\|_r^2 := \sum_k |k|^{2r} I_k \]
Natural general questions

1. Whether any solutions of the nonlinear problem are Periodic, Quasi Periodic or Almost Periodic
   This refers to the KAM theory for PDEs

2. Whether the action variables $I_k(z)$ are approximately conserved. This would imply upper bounds on growth of action variables, or on higher Sobolev norms
   This is in the realm of averaging theory for PDEs, including Birkhoff normal forms and Nekhoroshev stability

3. Whether there exist some solutions which exhibit a growing lower bound on the growth of the action variables
   These would be cascade orbits, related to the question of Arnold diffusion
Results

Theorem (C Garcia & WC (2012))
There exist Cantor families of periodic solutions (i.e. \( m = 1 \)) of the vortex filament equations (2) near the uniformly rotating solution \( v = 0 \)

Theorem (C Garcia, WC & CR Yang (in progress))
Given wavenumbers \( k_1, \ldots k_m \) there is a set \( a \in \mathcal{A} \) of full measure and an \( \varepsilon_0 = \varepsilon_0(a, k_1, \ldots k_m) \) such that for a Cantor set of amplitudes \( (b_1, \ldots b_m) \in B_{\varepsilon_0} \subseteq \mathbb{C}^m \) there exist QP solutions of (2) with \( m \)-many \( Q \) independent frequencies \( \Omega_j(b) \), of the form

\[
v(s, t) = \sum_{j=1}^{m} b_j e^{i k_j s} e^{i \Omega_j(b) t} + O(\varepsilon^2)
\]

Actually, these two theorems hold for any central configuration of vortices. The case of more complex configurations of near-vertical vortices is part of our future research program.
Hamiltonian PDEs

- Flow in phase space, where \( z \in \mathcal{H} \) a Hilbert space with inner product \( \langle X, Y \rangle_\mathcal{H} \),

\[
\partial_t z = J \text{grad}_z H(z), \quad z(x, 0) = z^0(x),
\]

(5)

- Symplectic form

\[
\omega(X, Y) = \langle X, J^{-1}Y \rangle_\mathcal{H}, \quad J^T = -J.
\]

- The flow \( z(x, t) = \varphi_t(z^0(x)), \) defined for \( z \in \mathcal{H}_0 \subseteq \mathcal{H} \)

- Theorem

The flow of (5) preserves the Hamiltonian function:

\[
H(\varphi_t(z)) = H(z), \quad z \in \mathcal{H}_0
\]

Proof: \( \frac{d}{dt}H(\varphi_t(z)) = \langle \text{grad}_z H, \dot{z} \rangle = \langle \text{grad}_z H, J \text{grad}_z H(z) \rangle = 0 \).
Hamiltonian PDEs: examples

- **Nonlinear Schrödinger equation**

- **Nonlinear wave equation**

- **Korteweg – de Vries equations**
  KAM tori: Kappeler & Pöschel (2003) ...

- **Euler’s equations for free surface water waves**
  Hamiltonian given by Zakharov (1968)

- **Vortex sheets**
Invariant tori - linear theory

- The tangent space approximation for the mapping $S$
  Linearize at $S$, set $\delta S = Z$ and use the self-adjoint form

$$
\Omega \cdot J^{-1} \partial_0 Z - \partial^2_\zeta H(S)Z = F
$$

(8)

Eigenvalues $\lambda$ of the LHS are the small divisors.

Analysis: Spectral properties of the linear operator are related to Anderson localization.

Treated by resolvent expansion methods developed by Fröhlich & Spencer, WC & Wayne, Bourgain, Berti & Bolle
Small divisors

- Frequencies of the linearized flow are \( \omega_k = \pm k \sqrt{k^2 + \frac{2}{a^2}} \).

Equation (8) is a small divisor problem. The linearized operator for a solution with \( m \) temporal quasi-periods \( \Omega = (\Omega_1, \ldots, \Omega_m) \in \mathbb{R}^m \) has eigenvalues

\[
\lambda_{jk}^\pm := k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega \cdot j)^2 + \frac{1}{a^4}}
\]

Proposition (small divisors)

For generic \( \Omega \) the eigenvalues \( \lambda_{jk}^- \) accumulate at \( \lambda = 0 \). For a set of full measure of \( \Omega \) the eigenvalues satisfy a diophantine estimate

\[
|\lambda_{jk}^-| \geq \frac{\gamma}{(|j| + |k|^2)^m + 1/2^+}
\]
A variational formulation for resonant invariant tori

- Mapping of a torus $S(\theta) : \mathbb{T}^m_\theta \mapsto \mathcal{H}$, which is flow invariant

$$S(\theta + t\Omega) = \varphi_t(S(\theta))$$

Frequency vector $\Omega \in \mathbb{R}^m$

- This implies that

$$\Omega \cdot \partial_\theta S = J \text{ grad}_z H(S) . \tag{9}$$

- Rewriting (9) in self-adjoint form

$$J^{-1}\Omega \cdot \partial_\theta S - \text{ grad}_z H(S) = 0 . \tag{10}$$

Recall: Problem of KAM tori: Solve (10) for $(S(\theta), \Omega)$, generally a small divisor problem, which in the present context also has a null space.
Space of torus mappings

Consider the space of mappings $S \in \mathcal{X} := \{S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}\}$

- Define average action functionals

$$\bar{I}_j(S) = \frac{1}{2} \int_{\mathbb{T}^m} \langle S, J^{-1} \partial_{\theta_j} S \rangle \, d\theta$$

$$\delta S \bar{I}_j = J^{-1} \partial_{\theta_j} S$$

The moment map for mappings

- The average Hamiltonian

$$\bar{H}(S) = \int_{\mathbb{T}^m} H(S(\theta)) \, d\theta$$

$$\delta S \bar{H} = \nabla_S H(S)$$
A variational formulation

Consider the subvariety of $\mathcal{X}$ defined by fixed actions

$$\mathcal{M}_a = \{ S \in \mathcal{X} : \bar{I}_1(S) = a_1, \ldots, \bar{I}_m(S) = a_m \} \subseteq \mathcal{X}$$

**Variational principle:** critical points of $\overline{H}(S)$ on $\mathcal{M}_a$ correspond to solutions of equation (7), with Lagrange multiplier $\Omega$.

**NB:** All of $\overline{H}(S)$, $\bar{I}_j(S)$ and $\mathcal{M}_a$ are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\theta) \mapsto S(\theta + \alpha)$, $\alpha \in \mathbb{T}^m$. 
The linearized vortex filament equations

Illustrate this with the linearized vortex filament equations

- The quadratic Hamiltonian

$$H^{(2)} = \int_0^{2\pi} \frac{1}{2} \left[ (\partial_s X)^2 + (\partial_s Y)^2 + \frac{2}{a^2} X^2 \right] ds$$

with frequencies

$$\omega_k = \pm k \sqrt{k^2 + (2/a^2)}$$

- Linearized equations for an invariant torus

$$\Omega \cdot \partial_\theta X = \text{grad}_Y H^{(2)} = -\partial_s^2 Y$$

$$\Omega \cdot \partial_\theta Y = -\text{grad}_X H^{(2)} = \partial_s^2 X - \frac{2}{a^2} X$$

- Fourier representation of torus mappings

$$S(x, \theta) = \sum_{k \in \mathbb{Z}} S_k(\theta) e^{ikx} = \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}^m} S_{jk} e^{ij \cdot \theta} e^{ikx}$$

Eigenvalues

$$\lambda_{jk}^\pm(\Omega) = k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega \cdot j)^2 + \frac{1}{a^4}}$$
Null space

- Choose \((\omega_{k1}, \ldots, \omega_{km})\) linear frequencies, and a frequency vector \(\Omega^0 = (\Omega^0_1, \ldots, \Omega^0_m)\) solving the resonance relations

\[
\lambda_{jk}^{-1}(\Omega^0) = 0.
\]

- This identifies a null eigenspace in the space of mappings

\[
X_1 \subseteq X.
\]

Proposition

\(X_1 \subseteq X\) is even dimensional; \(\dim(X_1) = 2M \geq 2m\). It is possibly infinite dimensional

- Nonresonant case: \(M = m\).
- Resonant case: \(M > m\).
Lyapunov - Schmidt decomposition

- Decompose $\mathcal{X} = \{ S : \mathbb{T}^m \to M \} = \mathcal{X}_1 \oplus \mathcal{X}_2 = Q\mathcal{X} \oplus P\mathcal{X}$.

- Equation (7) is equivalent to

$$Q(J^{-1}\Omega \cdot \partial_\theta S - \text{grad}_z H(S)) = 0,$$

$$P(J^{-1}\Omega \cdot \partial_\theta S - \text{grad}_z H(S)) = 0.$$  \hspace{1cm} (11) \hspace{1cm} (12)

- Decompose the mappings $S = S_1 + S_2$ as well.

- Small divisor problem for $S_2 = S_2(S_1, \Omega)$, which one solves for $(S_1, \Omega) \in \mathcal{E}$ a Cantor set.
Variational problem reduced to a link

It remains to solve the Q-equation (11). This can be posed variationally (with analogy to Weinstein - Moser theory).

- Define

\[
\begin{align*}
\bar{I}_j^1(S_1) &= \bar{I}_j(S_1 + S_2(S_1, \Omega)) \\
\bar{H}^1(S_1) &= \bar{H}(S_1 + S_2(S_1, \Omega)) \\
\mathcal{M}_a^1 &= \{ S_1 \in \mathcal{X}_1 : \bar{I}_j^1(S_1) = a_j, j = 1 \ldots m \}
\end{align*}
\]

- Critical points of $\bar{H}^1(S_1)$ on $\mathcal{M}_a^1$ are solutions of (11) with action vector $a$. 
equivariant Morse – Bott theory

The group $T^m$ acts on $M^1_a$ leaving $\overline{H}^1(S_1)$ invariant.

One seeks critical $T^m$ orbits.

Question: How many critical orbits of $\overline{H}^1$ on $M^1_a$?
Depends upon its topology.

Conjecture (a reasonable guess)

For given $a$ there exist integers $p_1, \ldots, p_m$ such that $\sum_j p_j = M$ and

$$M^1_a \simeq \bigotimes_{j=1}^m S^{2p_j-1}$$
Morse – Bott theory

Check this fact, in endpoint cases.

- Periodic orbits $m = 1$, resonant case $M > 1$.

$$\mathcal{M}_{a}^{1} \simeq S^{2M-1}, \quad \mathcal{M}_{a}^{1}/\mathbb{T}^{1} \simeq \mathbb{C}P_{w}(M - 1)$$

This restates the estimate of Weinstein and Moser

$$\# \{ \text{critical } \mathbb{T}^{1} \text{ orbits} \} \geq M$$
Morse – Bott theory

- Nonresonant quasi-periodic orbits $m = M$.

\[ \mathcal{M}_a^1 \simeq \bigotimes_{j=1}^M S^1, \quad \mathcal{M}_a^1 / \mathbb{T}^m \simeq \text{a point} \]

This corresponds to a Lagrangian KAM torus in Percival’s variational principle.

- The case $m = 2 \leq M$ occurs in the problem of doubly periodic traveling wave patterns on the surface of water.

\[ \mathcal{M}_a^1 \simeq S^{2p-1} \otimes S^{2(M-p)-1} \]
Theorem (Chaperon, Bosio & Meersmann (2006))

The topology of links $\mathcal{M}_a^1$ can be complex. There are cases in which

$$\mathcal{M}_a^1 \simeq \#_{\ell=1}^q (S^{2p_{\ell_1}-1} \otimes \cdots \otimes S^{2p_{\ell_k}-1}), \quad \sum_j p_{\ell_j} = M$$

Furthermore, there are more complex quantities than this.

**Proof:** combinatorics and cohomolological calculations.

Conjecture (revised opinion)

The number of distinct critical $\mathbb{T}^m$ orbits of $\overline{H}^1$ on $\mathcal{M}_a^1$ is bounded below:

$$\#\{\text{critical orbits of } \overline{H}^1 \text{ on } \mathcal{M}_a^1\} \geq (M - m + 1).$$
Thank you