Stochastic collocation methods for Stochastic differential equations driven by white noise

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Review on numerical methods for SDEs

The spectral approximation - single element
  Numerical results for geometric Brownian motion
  Burgers equation with multiplicative noise

The spectral approximation - multi-element
  the Smolyak Sparse grid method
  Burgers equation with additive noise

Comparison between single-element and multi-element approximations
  Convergence in time
  Convergence in $n$

Application to stochastic piston problem
  Verification
  Stochastic collocation method
  Quasi-Monte Carlo method
Consider a mean-square well-posed Stratonovich SDE

\[
dX = b(t, X)dt + \sigma(t, X) \circ dW(t), \quad t \in (0, T] \quad X(0) = X0. \quad (1)
\]

Two typical ways to solve:

- **Wong-Zakai approximation**: \( t \in [t_i, t_{i+1}) \),
  \[
  W(t) \approx W^n(t) = W(t_i) + (W(t_{i+1}) - W(t_i))\frac{t - t_i}{t_{i+1} - t_i}.
  \]

- **discretization in time first**
  \[
dX = \left( b(t, X) + \frac{1}{2}\sigma\sigma_x(t, X) \right) dt + \sigma(t, X)dW(t), \quad X(0) = X0, \quad (2)
  \]

Forward Euler scheme, e.g.

\[
\tilde{X}(t_{i+1}) = \tilde{X}(t_i) + (b(t_i, \tilde{X}(t_i)) + \frac{1}{2}\sigma(t_i, \tilde{X}(t_i))\sigma_x(t_i, \tilde{X}(t_i))(t_{i+1} - t_i)
  + \sigma(t_i, \tilde{X}(t_i))(W(t_{i+1}) - W(t_i)),
\]
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Replace $W(t_{i+1}) - W(t_i)$ with $\sqrt{t_{i+1} - t_i} \xi_i, \xi_i \sim \mathcal{N}(0, 1)$ (i.i.d.)
Solve the resulting equation

- **sparse grid** methods: Gerstner et al [1](1998) increasing dimensionality

- Need more appropriate approximations of Brownian motion if higher order methods are preferred.
- Lévy-Ciesielski (spectral) approximation (a.k.a Karhunen-Loeve)

\[
W(t) \approx W^n(t) = \sum_{i=1}^{n} \int_{0}^{t} m_i(s) \, ds \xi_i, t \in [0, T]. \tag{3}
\]
Algorithm (stochastic collocation method, single element)

1) Approximate the equation (1) with

\[
  d\tilde{X} = b(t, \tilde{X})dt + \sigma(t, \tilde{X}) \sum_{i=1}^{n} m_i(t)\xi_i dt, \quad \tilde{X}(0) = X_0, \quad (4)
\]

where \( \{m_k(s)\}_{l \geq 1} : CONS \text{ in } L^2([0, T]); \ \xi_k \sim \mathcal{N}(0, 1) \text{ i.i.d.} \)

2) Choose proper time discretization scheme and the stochastic collocation points to solve (4).

3) Compute the moments of the solution using the chosen quadrature rule in 2).
The collocation method for (1): for $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$d\tilde{X}(t, x_\alpha) = b(t, \tilde{X}(t, x_\alpha)) \, dt + \sigma(t, \tilde{X}(t, x_\alpha)) \sum_{k=1}^{n} m_k(t) \xi_k(\alpha_k) \, dt. \tag{5}$$

$x_\alpha = (\xi_k(\alpha_1), \cdots, \xi_n(\alpha_n))$, $\xi_k(\alpha_k) = x_{\alpha_k}$: tensor-product or sparse grids based on 1D Gauss-Hermite quadrature.

Particular choice of basis:

$$m_1(t) = \frac{1}{\sqrt{T}}, \quad m_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{(k-1)t}{T}\right) \quad (k \geq 2).$$

Example: taking $n = 1$ and small $T$, gives

$$d\tilde{X}(t, x_j) = b(t, \tilde{X}(t, x_j)) \, dt + \sigma(t, \tilde{X}(t, x_j)) \frac{x_j}{\sqrt{T}} \, dt. \tag{6}$$

Moments:

$$Ef(X(T)) \approx \sum_{i=1}^{N} f(\tilde{X}(T, x_j)) w_j.$$
A stochastic differential equation (SDE) example is given:

\[ dX = \lambda X dt + \mu X \circ dW(t), \quad X(0) = 1, \quad (7) \]

where \( X(t) = \exp(\lambda t + \mu W(t)) \), and the expected value of the SDE is given by 

\[ E[X(t)]^k = \exp(\lambda kt + k^2 \mu^2 t/2), \quad k = 1, 2, \ldots. \]

The figure illustrates the relative mean error and the relative second-order moment error for different parameter settings and time steps, demonstrating the performance of the stochastic collocation plus Crank-Nicolson scheme in time for the given SDE. The figure shows the slope of the error versus the polynomial degree, indicating convergence and accuracy of the numerical methods.
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Numerical results for geometric Brownian motion
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\begin{align*}
  u_t + (u + \sigma \dot{w}) \circ u_x &= \nu u_{xx}, \quad x \in (0, 2\pi), \quad \nu > 0. \\
  u(x, t) &= 2\nu \frac{e^{-\nu t} \sin(x - \sigma w(t))}{2 + e^{-\nu t} \cos(x - \sigma w(t))}.
\end{align*} (8)

Figure: Stochastic collocation, Fourier collocation + Crank Nicolson-Leap frog, for (8): $T=1$, $n=1$, $\nu = 1$, $\sigma = 0.1$. $\|Eu^k - Eu^k_N\|_\infty / \|Eu^k\|_\infty$
Remark

- Convergence of Step 1, Algorithm 2.1: Sussmann [7] (1978);
- Convergence of collocation is analogue to the spectral collocation methods in physical space.
- It often suffices to use just a few random variables at every step of the spectral approximation of Brownian motion if the solution is smooth enough.
- When several random variables are used, the stochastic collocation method by Xiu et al. [8] (2005) works.
Full tensor product ( $Q^1_N f - 1D$ quadrature) :

$$(Q^1_{i_1} \otimes \cdots \otimes Q^1_{i_d}) f = \sum_{\alpha_1=1}^{n_{i_1}} \cdots \sum_{\alpha_d=1}^{n_{i_d}} f(\alpha_1 x_{i_1}, \cdots, \alpha_d x_{i_d}) w_{\alpha_1} \cdots w_{\alpha_d}.$$ 

Smolyak sparse grid quadrature rule [6] (1963) :

$$Q^d_q f = \sum_{d \leq |i| \leq q} (U_{i_1} \otimes \cdots \otimes U_{i_d}) f,$$  

(9)

$$U^1_k = (Q^1_k - Q^1_{k-1}) f, \quad Q^1_0 f := 0.$$ 

where $\mathbf{i} = (i_1, i_2, \cdots, i_d)$ $i_k \geq 1$, $|\mathbf{i}| = i_1 + i_2 + \cdots + i_d$. 

The level of the sparse grid: $L =: q - d$. 


Spectral expansion over the interval $[0,T]$: 

\begin{align*}
0 & \delta t & T \\
\end{align*}

Local spectral approximation: $[k\Delta, (k + 1)\Delta]$: 

\begin{align*}
0 & \delta t & \Delta & 2\Delta & \cdots & \cdots & T = K\Delta \\
\end{align*}

◊ Algorithm of multi-element coincides with Algorithm of single-element if $K = 1$. 

H. Zhang et al.  Stochastic collocation methods for SDEs
Algorithm (stochastic collocation method, multi-element)

Partition: $0 = t_0 < \Delta \cdots < i\Delta < \cdots < K\Delta = T$.

1) Approximate the equation (1) with

$$d\tilde{X} = b(t, \tilde{X}) dt + \sigma(t, \tilde{X}) \sum_{i=1}^{n} \sum_{k=0}^{K-1} m_i^{(k)}(t)\xi_i^{(k)} dt, \quad \tilde{X}(0) = X_0,$$

where $m_i^{(k)}(t)$, $(t \in [k\Delta, (k+1)\Delta])$ CONS in $L^2([k\Delta, (k+1)\Delta])$:

$$\xi_k^{(i)} \sim \mathcal{N}(0, 1) \text{ i.i.d.}$$

2) Generate the sparse grid points with dimension $nK$ and desired sparse grid level $L$.

3) Compute $E[\tilde{X}(t, \tilde{X}_n^{K-1})]^p$ using the sparse grid quadrature rule.
\[ u_t + uu_x = \nu u_{xx} + \sigma \dot{w}, \quad x \in (0, 2\pi), \quad \nu > 0. \quad (11) \]

\[ u(x, t) = 2\nu \frac{e^{-\nu t} \sin(x - \sigma \int_0^t w(s) \, ds)}{2 + e^{-\nu t} \cos(x - \sigma \int_0^t w(s) \, ds)} + \sigma w(t). \]

<table>
<thead>
<tr>
<th>( \Delta )</th>
<th>((N, n))</th>
<th>( \frac{|E u - E u_{N,n}|<em>\infty}{|E u|</em>\infty} )</th>
<th>( \frac{|E u^2 - E u_{N,n}^2|<em>\infty}{|E u^2|</em>\infty} )</th>
<th># paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((3, 2))</td>
<td>6.6092e-06</td>
<td>1.6505e-05</td>
<td>9</td>
</tr>
<tr>
<td>5.0e-01</td>
<td>((3, 2))</td>
<td>1.8122e-06</td>
<td>4.5257e-06</td>
<td>81</td>
</tr>
<tr>
<td>1.0e-01</td>
<td>((3^1, 2))</td>
<td>8.2715e-08</td>
<td>5.0009e-07</td>
<td>841 (^1)</td>
</tr>
</tbody>
</table>

Table: different stochastic collocation methods, Fourier collocation + Crank-Nicolson-Leap-Frog: \( T=1, \sigma = 0.1, \nu = 1. \)

convergence order \( \Delta^2 \) in first two moments.

\(^1\)Sparse grid with dimension \( nK = 2 \times 10 \), level \( L = 3 \).
**Table:** Test of Algorithm 2.1 (single-element spectral approximation) on the Burgers equation (11): Crank-Nicolson/Leap-Frog in time and Fourier collocation method in physical space with $\sigma = 0.1$, $\nu = 1$.

<table>
<thead>
<tr>
<th>$(N, nK)$</th>
<th>$\delta t$</th>
<th>$\frac{|Eu-Eu_{N,n}|<em>{\infty}}{|Eu|</em>{\infty}}$</th>
<th>order</th>
<th># grid points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 2$\times$1)</td>
<td>1.0e-3</td>
<td>6.6102e-06</td>
<td>$n^{-3.10}$</td>
<td>13</td>
</tr>
<tr>
<td>(3, 4$\times$1)</td>
<td>1.0e-3</td>
<td>7.7069e-07</td>
<td>$n^{-2.39}$</td>
<td>41</td>
</tr>
<tr>
<td>(4, 8$\times$1)</td>
<td>1.0e-4</td>
<td>1.4709e-07</td>
<td>$n^{-2.39}$</td>
<td>849</td>
</tr>
<tr>
<td>(4,16$\times$1)</td>
<td>1.0e-4</td>
<td>1.5464e-08</td>
<td>$n^{-3.25}$</td>
<td>6049</td>
</tr>
</tbody>
</table>
Table: Test of Algorithm 2.1 (single-element spectral approximation) on the Burgers equation (11): Crank-Nicolson/Leap-Frog in time and Fourier collocation method in physical space with $\sigma = 0.1$, $\nu = 1$.

<table>
<thead>
<tr>
<th>$(N, nK)$</th>
<th>$\delta t$</th>
<th>$\frac{| Eu^2 - Eu^2_{N,n} |<em>\infty}{| Eu^2 |</em>\infty}$</th>
<th>order</th>
<th># grid points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 2×1)</td>
<td>1.0e-3</td>
<td>1.6077e-05</td>
<td>–</td>
<td>13</td>
</tr>
<tr>
<td>(3, 4×1)</td>
<td>1.0e-3</td>
<td>2.7122e-06</td>
<td>$n^{-2.57}$</td>
<td>41</td>
</tr>
<tr>
<td>(4, 8×1)</td>
<td>1.0e-4</td>
<td>3.4121e-07</td>
<td>$n^{-2.99}$</td>
<td>849</td>
</tr>
<tr>
<td>(4,16×1)</td>
<td>1.0e-4</td>
<td>4.2995e-08</td>
<td>$n^{-2.99}$</td>
<td>6049</td>
</tr>
</tbody>
</table>

Conclusion: $\Delta^2$ versus $n^{-3}$ – spectral approximation shows faster convergence!
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Figure: piston with velocity perturbed by Brownian motion

- perturbation analysis, Lin et al [3]
- stochastic Euler equation

\[
\frac{\partial}{\partial \tau} \mathbf{V} + \frac{\partial}{\partial y} (f(\mathbf{V})) = g(\mathbf{V}) \ast \dot{\mathbf{W}}, \quad (12)
\]

'\ast\': the Stratonovich product; '\cdot\': the Ito product
\( \mathbf{V} = (v, \rho v, E)^T, \ g(\mathbf{V}) = (0, -\rho, -\rho v), \)
\( f(\mathbf{V}) = (\rho v, \rho v^2 + P, \nu(P + E)). \)

Splitting + WENO + SSP-RK2

\[
\frac{\partial}{\partial \tau} \mathbf{V}^{(1)} + \frac{\partial}{\partial y} (f(\mathbf{V}^{(1)})) = 0, \tag{13}
\]

with initial condition \( \mathbf{V}^{(1)}(\tau_n) = \mathbf{V}(\tau_n) \) and boundary condition then

\[
\frac{\partial}{\partial \tau} \mathbf{V}^{(2)} = g(\mathbf{V}^{(2)}) \ast \dot{\mathbf{W}}, \tag{14}
\]

with the initial condition \( \mathbf{V}^{(2)}(\tau_n) = \mathbf{V}^{(1)}(\tau_{n+1}) \) and no boundary condition.
Figure: Comparison between perturbation analysis and stochastic Euler equation
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Figure: Comparison between numerical results from Stratonovich-Euler equation (12) using direct Monte Carlo method and stochastic collocation method.

(a) $\epsilon = 0.02$

(b) $\epsilon = 0.05$
Figure: Comparison between numerical results from Stratonovich-Euler equation (12) using direct Monte Carlo method and stochastic collocation method.
Figure: Comparison between numerical results from Stratonovich-Euler equation (12) using direct Monte Carlo method and stochastic collocation method: $\epsilon = 0.5$. 
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Figure: Comparison between numerical results from Stratonovich-Euler equation (12) using direct Monte Carlo method and the quasi-Monte Carlo method-scrambled Sobol sequence: $\epsilon = 0.5$. $n \times \frac{T}{\Delta} = n \times K$

(a) Sobol sequence: 500 sample points (b) Sobol sequence: 1000 sample points
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**Figure:** Comparison between numerical results from Stratonovich-Euler equation (12) using direct Monte Carlo method and the quasi-Monte Carlo method-scrambled Halton sequence: $\epsilon = 0.5$. 

(a) Halton sequence: 500 sample points

(b) Halton sequence: 1000 sample points
Conclusion and Discussion

- Karhunen-Loeve expansion of Brownian motion, Wong-Zakai type approximation
- Spectral convergence in random space, smoothness required
- Sparse grid collocation; still curse of dimensionality; other choices of integration methods may apply: Quasi-Monte Carlo, ANOVA, et al.
- Stochastic piston problem, multi-element spectral approximations.

Future work

- Multiple Brownian motions, correction terms
- Nonlinear filtering problem: recursive algorithms
- SPDEs with spatial noise or spatial-temporal noises.
- Stochastic KdV equation
- SPDEs driven by fractional Brownian motion (colored)
Thank you!


