CONJUGATE UNSCENTED TRANSFORMATION – “OPTIMAL QUADRATURE”

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H. Bjornsson, S. Carn, K. Dean, M. Pavolonis, M. Ripepe, P. Webley
Stochastic System: $\dot{x} = f(\Theta, x) + g(\Theta, x)\eta(t)$

- **Source of Uncertainties**: system parameters, initial conditions, input to the system, modeling error.

- Robust modeling of the propagation of these uncertainties is important to accurately quantify the uncertainty in the solution at any future time.

*Figure: State and pdf transition.*
**Approximate Solution to exact problem:** Kolmogorov Equation, Monte Carlo Methods, generalized Polynomial Chaos (gPC).

**Exact solution to approximate problem:** Gaussian Closure, Equivalent Linearization, and Stochastic Averaging.

All these methods involve the evaluation of expectation integrals:

\[
E[f(x)] = \int \int \cdots \int w(x)f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)
\]
Approximate Solution to exact problem: Kolmogorov Equation, Monte Carlo Methods, generalized Polynomial Chaos (gPC).

Exact solution to approximate problem: Gaussian Closure, Equivalent Linearization, and Stochastic Averaging.

All these methods involves the evaluation of expectation integrals:

\[ E[f(x)] = \int \int \cdots \int w(x)f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \]
**INTRODUCTION**

**REVIEW OF QUADRATURE RULES**

**METHODS**

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules
INTRODUCTION

REVIEW OF QUADRATURE RULES

METHODS

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules

\[ E[f(\mathbf{x})] = \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}_i) \]

- Involves the random sampling of the pdf \( w(\mathbf{x}) \).
- Error decrease as a factor of \( \frac{1}{\sqrt{n}} \).
**Introduction**

Review of Quadrature Rules

**Methods**

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules


\[
E[f(x)] = \sum_{i=1}^{n} w_i f(x_i)
\]

- The weights \(w_i\) and points \(x_i\) are deterministic.

- 1-D integrals: \(m\) quadrature points are required to reproduce the expectation integrals involving \(2m - 1\) degree polynomial functions.

- N-D integrals: Tensor product leads to exponential growth of points \((m^N)\).

- Nested quadratures: Clenshaw-Curtis.
**Methods**

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules

\[ E[f(x)] = \sum_{i=1}^{n} w_i f(x_i) \]

- The weights \( w_i \) and points \( x_i \) are deterministic.
- The *curse of dimensionality* involved in Gaussian Quadratures is avoided by considering a sparse tensor product of 1-Dimensional quadrature rules.
- Example: Smolyak Quadratures.
- Some weights can be negative.
**INTRODUCTION**

**REVIEW OF QUADRATURE RULES**

**METHODS**

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules

\[ E[f(x)] = \sum_{i=1}^{n} w_i f(x_i) \]

- The weights \( w_i \) and points \( x_i \) are deterministic.\(^1\)

- \( 2N + 1 \) points are required to integrate all polynomials up to degree 3.

\[ \text{Number of points} \]

\[ \text{Method} \quad \text{Dimension} \]

<table>
<thead>
<tr>
<th>Method</th>
<th>Dimension</th>
<th>Number of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>GH2</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>SmolyakGH</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>UT2</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>SmolyakGH</td>
<td>6</td>
<td>40</td>
</tr>
</tbody>
</table>

**Figure:** Comparison of 3rd degree rules

**Methods**

- Monte Carlo Methods
- Gaussian Quadrature
- Sparse Grid Quadrature
- Unscented Transform
- Minimal Cubature rules

\[ E[f(x)] = \sum_{i=1}^{n} w_i f(x_i) \]

- The weights \( w_i \) and points \( x_i \) are deterministic.
- Minimum cubature rules try to evaluate the expectation integrals with as few points as possible.
- A extensive work was carried out by Stroud\(^1\), where a number of minimal cubature rules of various order and dimension are documented.


**Table:** Minimal Cubature Rules for Gaussian Integrals

<table>
<thead>
<tr>
<th>Degree</th>
<th>Points</th>
<th>Valid for</th>
<th>All positive weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$2N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>$2^N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$N^2 + N + 2$</td>
<td>$2 \leq N \leq 7$</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$2^N + 2N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$2N^2 + 1$</td>
<td>all dimensions</td>
<td>Only for $N \leq 3$</td>
</tr>
<tr>
<td>7</td>
<td>$2^N + 2N^2 + 1$</td>
<td>$N = 3, 4, 6, 7$</td>
<td>Only for $N \neq 7$</td>
</tr>
<tr>
<td>7</td>
<td>$(4N^3 + 8N + 3)/3$</td>
<td>$N = 3, 4, 5, 6$</td>
<td>No</td>
</tr>
<tr>
<td>9</td>
<td>$(2N^4 - 4N^3 + 22N^2 - 8N + 3)/3$</td>
<td>$N = 3, 4, 5, 6$</td>
<td>No</td>
</tr>
</tbody>
</table>

$N \rightarrow$ is the dimension of the integral

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*b* No solution exists for other dimensions

*c* No solution/Imaginary solution for other dimensions
**Cubature Rule:** \( E[f(x)] = \int \int \cdots \int w(x)f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \)

- Most of the non-product cubature rule possess certain similarities:
  - exploit the symmetry of the pdf, assume a structure for the cubature points, solve a system of nonlinear equations.

- **Drawbacks:** inconsistency of the set of nonlinear equations, the presence of negative weights/complex roots and the inability to provide a generalized solution that works for any dimension.
**Introduction**

**Objective**

**Cubature Rule:**

\[ E[f(x)] = \int \int \cdots \int w(x)f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \]

- Most of the non-product cubature rule possess certain similarities:
  - exploit the symmetry of the pdf, assume a structure for the cubature points, solve a system of nonlinear equations.
- **Drawbacks:** inconsistency of the set of nonlinear equations, the presence of negative weights/complex roots and the inability to provide a generalized solution that works for any dimension.

**Primary Objective**

- To find a fully symmetric reduced sigma/cubature point set with all positive weights that sum up to one and that is equivalent to the Gaussian quadrature product rule of same order.
3rd order rule with $2N + 1$ points.

Can integrate all odd degree monomials due to symmetry of the points chosen.

This rule cannot be extended to higher degree since points constrained to lie on principal axes cannot capture any cross moment.

The method provides the insight: ‘To choose additional axes that are able to capture higher order moments’

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2Julier, Uhlmann, and Durrant-Whyte, “A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators”.

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**Unscented Transform**

$X_0 = (0, \ldots, 0)$

$W_0 = \frac{\kappa}{(N + \kappa)}$

$X_i = \sqrt{(N + \kappa)}I_i$

$W_i = \frac{1}{2(N + \kappa)}$

$X_{i+N} = -\sqrt{(N + \kappa)}I_i$

$W_{i+N} = \frac{1}{2(N + \kappa)}$

**Figure:** Selection of points in 2D
Moment Constraint Equations (MCEs):

\[
\sum_{i=1}^{n} w_i \{ x(i,1)^n x(i,2)^{n_2} \cdots x(i,N)^{n_N} \} = E[ x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} ]
\]

A Fully Symmetric Set

A set of points is called fully symmetric if it is closed under all coordinate and sign permutations.

- For example consider the set \( X = \{ x_1, x_2, \cdots \} \) to be a fully symmetric set and if \( x_i = [a, b]^T \in X \), then
  \[
  \{ [b, a]^T, [−a, b]^T, [a, −b]^T, \]
  \[
  [−a, −b]^T, [−b, a]^T, [b, −a]^T, [−b, −a]^T \} \in X.
  \]
σ: Represents the *Principal axes* (or orthogonal axis) in the cartesian coordinate space. Each point on the principal axis is denoted as $\sigma_i$

**TABLE:** Fully Symmetric set of points

<table>
<thead>
<tr>
<th>Type</th>
<th>Sample Point</th>
<th>No. of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>$(1,0,0,\cdots,0)$</td>
<td>$2N$</td>
</tr>
<tr>
<td>$\epsilon^M$</td>
<td>$(1,1,\cdots,1,0,0,\cdots,0)$</td>
<td>$2^M \binom{N}{M}$</td>
</tr>
<tr>
<td>$s^M(h)$</td>
<td>$(h,1,1,\cdots,1)$</td>
<td>$N2^M$</td>
</tr>
</tbody>
</table>
$c^M$: Represents the $M^{th}$ Conjugate axes (fully symmetric set) and the corresponding points are enumerated as $c^M_i$.

**Table:** Fully Symmetric set of points

<table>
<thead>
<tr>
<th>Type</th>
<th>Sample Point</th>
<th>No. of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>$(1, 0, 0, \cdots, 0)$</td>
<td>$2N$</td>
</tr>
<tr>
<td>$c^M$</td>
<td>$(1, 1, \cdots, 1, 0, 0, \cdots, 0)$</td>
<td>$2^M \binom{N}{M}$</td>
</tr>
<tr>
<td>$s^N(h)$</td>
<td>$(h, 1, 1, \cdots, 1)$</td>
<td>$N 2^N$</td>
</tr>
</tbody>
</table>
**Conjugate Unscented Transformation**

**Definitions**

\( s^N(h) \): Represents the *Scaled Conjugate axes* and the points are denoted as \( s^N_i(h) \). The parameter \( h \) is a scaling factor.

**Table:** Fully Symmetric set of points

<table>
<thead>
<tr>
<th>Type</th>
<th>Sample Point</th>
<th>No. of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>( (1,0,0,\cdots,0) )</td>
<td>( 2N )</td>
</tr>
<tr>
<td>( \epsilon^M )</td>
<td>( (1,1,\cdots,1,0,0,\cdots,0) )</td>
<td>( 2^M \binom{N}{M} )</td>
</tr>
<tr>
<td>( s^N(h) )</td>
<td>( (h,1,1,\cdots,1) )</td>
<td>( N2^N )</td>
</tr>
</tbody>
</table>
CONJUGATE UNSCENTED TRANSFORMATION

Graphical visualization of the Points/axes

\[
\begin{align*}
E[X_i^2] \\
E[X_i^4] \\
E[X_i^2 X_j^2]
\end{align*}
\]

Selection of axes and points

\[
\begin{align*}
2r_1^2w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4w_1 + 2^N r_2^4w_2 &= 3 \\
2^N r_2^4w_2 &= 1
\end{align*}
\]

Moments of desired order

Moment Constraint Equations
CONJUGATE UNSCENTED TRANSFORMATION  
CUBATURE POINTS OF $2^{nd}/3^{rd}$ ORDER

**GAUSSIAN PDF**

The moments up to $2^{nd}$ order are:

$$\sum_{i=1}^{n} w_i = 1, \quad E[x_i^2] = 1$$

- Select a set of points on $\sigma$ at a distance of $r_1$ and weight $w_1$.
- MCEs are:
  $$2Nw_1 = 1, \quad 2r_1^2w_1 = 1$$
- The solution is:
  $$w_1 = \frac{1}{2N}, \quad r_1 = \sqrt{N}$$
- Choosing $r_1 = N + \kappa$ gives the UT

**UNIFORM PDF**

The moments up to $2^{nd}$ order are:

$$\sum_{i=1}^{n} w_i = 1, \quad E[x_i^2] = 1/3$$

- Select a set of points on $\sigma$ at a distance of $r_1$ and weight $w_1$.
- The solution is:
  $$w_1 = \frac{1}{2N}, \quad r_1 = \sqrt{\frac{N}{3}}$$
- Above dimension 3, the points go out of the support $\Omega = [-1, 1]$
- Alternatively, choose $c^N$ instead of $\sigma$ for which the solution of MCEs are:
  $$w_1 = \frac{1}{2N}, \quad r_1 = \frac{1}{\sqrt{3}}$$

---

## Table: CUT4

<table>
<thead>
<tr>
<th>Position</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$X_i = r_1 \sigma_i$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$W_i = w_1$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$X_i + 2N = r_2 c_i^N$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$W_i + 2N = w_2$</td>
</tr>
<tr>
<td>Central weight</td>
<td>$X_0 = 0$</td>
</tr>
<tr>
<td>Central weight</td>
<td>$W_0 = w_0$</td>
</tr>
</tbody>
</table>

$n = 2N + 2^N (+1)$

## Table: CUT6, ($N \leq 6$)

<table>
<thead>
<tr>
<th>Position</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$X_i = r_1 \sigma_i$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$W_i = w_1$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$X_i + 2N = r_2 c_i^N$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N$</td>
<td>$W_i + 2N = w_2$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N(N - 1)$</td>
<td>$X_i + 2N + 2^N = r_3 c_i^2$</td>
</tr>
<tr>
<td>$1 \leq i \leq 2N(N - 1)$</td>
<td>$W_i + 2N + 2^N = w_3$</td>
</tr>
<tr>
<td>Central weight</td>
<td>$X_0 = 0$</td>
</tr>
<tr>
<td>Central weight</td>
<td>$W_0 = w_0$</td>
</tr>
</tbody>
</table>

$n = 2N^2 + 2^N + 1$

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**Conjugate Unscented Transformation**

Cubature Points for 4th/5th and 6th/7th Degree
### Table: Sigma Points for CUT8, \(2 \leq N \leq 6\)

<table>
<thead>
<tr>
<th>Position</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 \leq i \leq 2N)</td>
<td>(X_i = r_1 \sigma_i)</td>
</tr>
<tr>
<td>(1 \leq i \leq 2N)</td>
<td>(X_{i+2N} = r_2 c_i^N)</td>
</tr>
<tr>
<td>(1 \leq i \leq 2N(N-1))</td>
<td>(X_{i+2N+2N} = r_3 c_i^2)</td>
</tr>
<tr>
<td>(1 \leq i \leq 2^N)</td>
<td>(X_{i+2N+2N+2N(N-1)} = r_4 c_i^N)</td>
</tr>
<tr>
<td>(1 \leq i \leq N_1)</td>
<td>(X_{i+2N+2N+2N(N-1)+2N} = r_5 c_i^3)</td>
</tr>
<tr>
<td>(1 \leq i \leq N 2^N)</td>
<td>(X_{i+2N+2N+2N(N-1)+2N+N_1} = r_6 c_i^N)</td>
</tr>
</tbody>
</table>

Central weight | \(X_0 = 0\) | \(W_0 = w_0\)

\[n = 2N + 2^N + 2N(N-1) + N_1 + 2^N + N 2^N + 1, \quad \{N_1 = 4N(N-1)(N-2)/3\}\]
### Minimal Rules for Gaussian Integrals

<table>
<thead>
<tr>
<th>Degree</th>
<th>Points</th>
<th>Valid for</th>
<th>All positive weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$2N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>$2^N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$N^2 + N + 2$</td>
<td>$2 \leq N \leq 7^a$</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$2^N + 2N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>$2N^2 + 1$</td>
<td>all dimensions</td>
<td>Only for $N \leq 3$</td>
</tr>
<tr>
<td>7</td>
<td>$2N^2 + 2N^2 + 1$</td>
<td>$N = 3, 4, 6, 7^b$</td>
<td>Only for $N \neq 7$</td>
</tr>
<tr>
<td>7</td>
<td>$(4N^3 + 8N + 3)/3$</td>
<td>$N = 3, 4, 5, 6$</td>
<td>No</td>
</tr>
<tr>
<td>9</td>
<td>$(2N^4 - 4N^3 + 22N^2 - 8N + 3)/3$</td>
<td>$N = 3, 4, 5, 6$</td>
<td>No</td>
</tr>
</tbody>
</table>

$^a$No solution exists for other dimensions

$^b$No solution/Imaginary solution for other dimensions

### The Conjugate Unscented Transform

<table>
<thead>
<tr>
<th>Degree</th>
<th>Points</th>
<th>Valid for</th>
<th>All positive weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$2^N + 2N$</td>
<td>all dimensions</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>$2N^2 + 2N^2 + 1$</td>
<td>$N \leq 6$</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>$2N + 2^N + N_1 + 1$</td>
<td>$7 \leq N \leq 9$</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>$2N + 2^N + N_1 + 1$</td>
<td>$10 \leq N \leq 13$</td>
<td>No</td>
</tr>
<tr>
<td>9</td>
<td>$2N^2 + 2^N + 2N(N - 1) + N_1 + 2^N + N_2^N + 1$</td>
<td>$N \leq 10$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

$N_1 = 4N(N - 1)(N - 2)/3$
**Numerical Experiments**

**Example:** \( E \left[ (\sqrt{1 + x^T x})^\alpha \right] = \int (\sqrt{1 + x^T x})^\alpha \mathcal{N}(x: 0, 0.1) \, dx \)

(a) Gauss-Hermite Convergence

(b) CUT Convergence

(c) Number of points required to achieve 0.5% Rel. error
Numerical Experiments
Re-entry Vehicle Dynamics

Re-entry Vehicle Dynamics

\[
\begin{align*}
\dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) \\
\dot{x}_3(t) &= D(t)x_3(t) + G(t)x_1(t) + v_1(t) \\
\dot{x}_4(t) &= D(t)x_4(t) + G(t)x_2(t) + v_2(t) \\
\dot{x}_5(t) &= v_3(t)
\end{align*}
\]

\[
D(t) = -\beta(t)\exp\left\{\frac{[R_0 - R(t)]}{H_0}\right\}V(t)
\]

\[
G(t) = -\frac{Gm_0}{R^3(t)}
\]

\[
\beta(t) = \beta_0\exp(x_5(t))
\]

\[
R(t) = \sqrt{x_1^2(t) + x_2^2(t)}
\]

\[
V(t) = \sqrt{x_3^2(t) + x_4^2(t)}
\]

\[
\begin{bmatrix}
6500.4 \\
349.14 \\
-1.8093 \\
-6.7967 \\
0.6932
\end{bmatrix},
\begin{bmatrix}
0.1 & 0 & 0 & 0 & 0 \\
0 & 0.1 & 0 & 0 & 0 \\
0 & 0 & 0.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Figure: Typical Trajectory

\[\mu_0 = \begin{bmatrix} 6500.4 \\ 349.14 \\ -1.8093 \\ -6.7967 \\ 0.6932 \end{bmatrix}, P_0 = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\]

---

Julier, Uhlmann, and Durrant-Whyte, “A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators”.

Figure: Convergence of MC Mean

Figure: % error in the mean of $x_1$

Figure: Convergence of MC Cov

Figure: % error in the mean of $x_3$
**Numerical Experiments**

**Re-entry Vehicle Dynamics**

**Figure:** % error in the var of $x_1$

**Figure:** % error in the var of $x_3$

**Table:** % Maximum Rel. error in Frobenius norms of Cov matrix

<table>
<thead>
<tr>
<th>method</th>
<th>CKF</th>
<th>UT</th>
<th>CUT4</th>
<th>CUT6</th>
<th>CUT8</th>
<th>GH5</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of points</td>
<td>10</td>
<td>11</td>
<td>42</td>
<td>83</td>
<td>355</td>
<td>3125</td>
<td>100000</td>
</tr>
<tr>
<td>% error</td>
<td>11.5306</td>
<td>17.6092</td>
<td>1.9398</td>
<td>1.0396</td>
<td>0.9391</td>
<td>0.9369</td>
<td>0.2019</td>
</tr>
</tbody>
</table>
The BENT integral eruption column model was used to produce eruption column parameters (mass loading, column height, grain size distribution) given a specific atmospheric sounding and source conditions.

- BENT takes into consideration atmospheric (wind) conditions as given by atmospheric sounding data.
- Plume rise height is given as a function of volcanic source and environmental conditions.

The PUFF Lagrangian model was used to propagate ash parcels in a given wind field (NCEP Reanalysis).

- PUFF takes into account dry deposition as well as dispersion and advection.

Polynomial chaos quadrature (PCQ) was used to select sample points and weights in the uncertain input space of vent radius, vent velocity, mean particle size and particle size variance.
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**Table:** Eruption source parameters based on observations of Eyjafjallajökull volcano and information from other similar eruptions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value range</th>
<th>PDF</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vent radius, $b_0$, m</td>
<td>65-150</td>
<td>Uniform</td>
<td>Measured from radar image of summit vents</td>
</tr>
<tr>
<td>Vent velocity, $w_0$, m/s</td>
<td>Range: 45-124</td>
<td>Uniform</td>
<td>M. Ripepe, Geneva, Switzerland, 2010, presentation</td>
</tr>
<tr>
<td>Mean grain size, $Md_\phi$</td>
<td>2 boxcars: 1.5-2 and 3-5</td>
<td>Multi-Modal Uniform</td>
<td>Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian. A. Hoskuldsson, Iceland meeting 2010, presentation</td>
</tr>
<tr>
<td>$\sigma_\phi$</td>
<td>1.9 ± 0.6</td>
<td>Uniform</td>
<td>Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian.</td>
</tr>
</tbody>
</table>
Figure: Concentration (52N 13.5E) vs. Number of PUFF Particles
Numerical Experiments
Iceland Volcano (Eyjafjallajökull) Eruption

Puff Particle Count Runtime Dependence
Xeon X5560 (2.8GHz) 24GB Total Memory, -runHours 144

<table>
<thead>
<tr>
<th>nAsh</th>
<th>$10^5$</th>
<th>$5 \times 10^5$</th>
<th>$10^6$</th>
<th>$2 \times 10^6$</th>
<th>$4 \times 10^6$</th>
<th>$8 \times 10^6$</th>
<th>$10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conc.</td>
<td>$7.40 \times 10^{-5}$</td>
<td>$1.17 \times 10^{-4}$</td>
<td>$1.07 \times 10^{-4}$</td>
<td>$1.12 \times 10^{-4}$</td>
<td>$1.09 \times 10^{-4}$</td>
<td>$1.15 \times 10^{-4}$</td>
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</tr>
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</table>
Figure: $9^4$ Clenshaw Curtis Runs

Figure: 161 CUT Runs
Figure: 9^4 Clenshaw Curtis Runs

Figure: 161 CUT Runs
Comparing CUT with Clenshaw Curtis

Iceland Volcano (Eyjafjallajökull) Eruption

**Figure:** Mean

**Figure:** Standard Deviation
PROBABILITY OF ASH TOP HEIGHT
ICELAND VOLCANO (EYJAFJALLAJÖKULL) ERUPTION

Figure: $Pr(\text{Top Ash Height}) \geq 0\text{km}$

Figure: $Pr(\text{Top Ash Height}) \geq 2\text{km}$

Figure: $Pr(\text{Top Ash Height}) \geq 4\text{km}$

Figure: $Pr(\text{Top Ash Height}) \geq 6\text{km}$
PROBABILITY OF ASH TOP HEIGHT
ICELAND VOLCANO (EYJAFJALLAJÖKULL) ERUPTION

(a) 00 hrs

(b) 06 hrs

(c) 12 hrs

(d) 18 hrs
Efficient quadrature scheme is developed for the accurate determination of high dimension expectation integrals involving symmetric pdfs.

- non-product cubature rule.
- generalization for any pdf.

Numerical examples illustrates the effectiveness of the new quadrature scheme.

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This work is jointly supported through National Science Foundation under Awards No. CMMI- 1054759, CMMI-1131074 and AFOSR FA9550-11-1-0336.
**FILTERS USED**

- CKF - with 10 points
- UKF - with 11 points
- CUT6 - with 83 points
- Particle Filter (PF) - with 10000 samples

The initial condition uncertainty is

\[
x_0 = \begin{bmatrix} 1000 \text{ m} \\ 300 \text{ m/s} \\ 1000 \text{ m} \\ 0 \text{ m/s} \\ -\frac{3\pi}{180} \text{ rad/s} \end{bmatrix}, \quad P_{0/0} = \begin{bmatrix} 100 \text{ m}^2 & 0 & 0 & 0 & 0 \\ 0 & 10^2 \text{ m}^2/\text{s}^2 & 0 & 0 & 0 \\ 0 & 0 & 100 \text{ m}^2 & 0 & 0 \\ 0 & 0 & 0 & 10^2 \text{ m}^2/\text{s}^2 & 0 \\ 0 & 0 & 0 & 0 & 100 \times 10^{-6} \text{ rad}^2/\text{s}^2 \end{bmatrix}
\]
% 2-NORM $RMSE_{pos}$ VS FREQ OF MEASUREMENT

(a) Bearing std dev $\sigma_\theta = 0.1811^\circ$

(b) Bearing std dev $\sigma_\theta = 0.5435^\circ$

(c) Bearing std dev $\sigma_\theta = 0.90593^\circ$

(d) Bearing std dev $\sigma_\theta = 1.993^\circ$
% 2-NORM $\text{RMSE}_{pos}$ VS FREQ OF MEASUREMENT

(e) Bearing std dev $\sigma_\theta = 1.2683^\circ$

(f) Bearing std dev $\sigma_\theta = 1.6307^\circ$

(g) Bearing std dev $\sigma_\theta = 2.3554^\circ$

(h) Bearing std dev $\sigma_\theta = 2.7178^\circ$
Polar to Cartesian

\[ E \begin{bmatrix} x \\ y \end{bmatrix} = \int \int \begin{bmatrix} r \cos(\theta) \\ r \sin(\theta) \end{bmatrix} \mathcal{N} \left( \begin{bmatrix} r \\ \theta \end{bmatrix} : \begin{bmatrix} \mu_r \\ \mu_\theta \end{bmatrix}, \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix} \right) \, dr \, d\theta \]

\begin{align*}
\mu_r &= 1 m \\
\sigma_r &= 0.02^2 m^2 \\
\mu_\theta &= 90 \text{ deg} \\
\sigma_\theta &= 15^2 \text{ deg}^2
\end{align*}

\begin{align*}
\mu_r &= 50 m \\
\sigma_r &= 0.02^2 m^2 \\
\mu_\theta &= 0 \text{ deg} \\
\sigma_\theta &= 30^2 \text{ deg}^2
\end{align*}