The Optimal Uncertainty Quantification framework applied to the seismic safety assessment of a truss structure

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The UQ challenge in the certification context

\[ G : \chi \longrightarrow \mathbb{R} \]
\[ X \longrightarrow G(X) \]
\[ \mathbb{P} \in \mathcal{M}(\chi) \]

You want to certify that

\[ \mathbb{P}[G(X) \geq a] \leq \epsilon \]

Problem

- You don’t know \( G \).
- You don’t know \( \mathbb{P} \).
The UQ challenge in the certification context

\[ G : \chi \rightarrow \mathbb{R} \]

\[ X \rightarrow G(X) \]

You want to certify that

\[ \mathbb{P}\left[G(X) \geq a\right] \leq \epsilon \]

You only know

\[ (G, \mathbb{P}) \in \mathcal{A} \]

\[ \mathcal{A} \subset \left\{ (f, \mu) \mid f : \chi \rightarrow \mathbb{R}, \mu \in \mathcal{P}(\chi) \right\} \]
Optimal bounds on $\mathbb{P}[G(X) \geq a]$

$U(\mathcal{A}) := \sup_{(f,\mu) \in \mathcal{A}} \mu[f(X) \geq a]$

$L(\mathcal{A}) := \inf_{(f,\mu) \in \mathcal{A}} \mu[f(X) \geq a]$

$L(\mathcal{A}) \leq \mathbb{P}[G(X) \geq a] \leq U(\mathcal{A})$

$U(\mathcal{A}) \leq \epsilon$: Safe even in worst case.

$\epsilon < L(\mathcal{A})$: Unsafe even in best case.

$L(\mathcal{A}) \leq \epsilon < U(\mathcal{A})$: Cannot decide.

Unsafe due to lack of information.
OUQ problems are a priori infinite dimensional, non-convex and highly constrained.

But as in linear programming, OUQ problems reduce to searches over finite dimensional families of extremal scenarios of $A$.

The dimension of the reduced problem is proportional to the number of probabilistic inequalities that describe $A$. 

\[ \text{ex}(A) \]
A simple example

What is the least upper bound on \( \mathbb{P}[X \geq a] \)?

If all you know is \( \mathbb{E}[X] \leq m \)

and \( \mathbb{P}[0 \leq X \leq 1] = 1 \)?

Answer

\[
\sup_{\mu \in \mathcal{A}} \mu \left[ X \geq a \right]
\]

\[\mathcal{A} = \{ \mu \in \mathcal{M}([0,1]) \mid \mathbb{E}_\mu[X] \leq m \}\]
You are given one pound of play-doh. How much mass can you put above \( a \) while keeping the seesaw balanced around \( m \)?

**Answer**

Markov’s inequality

\[
\sup_{\mu \in \mathcal{A}} \mu \left[ X \geq a \right] = \frac{m}{a}
\]

\[\mathcal{A} = \{ \mu \in \mathcal{M}([0, 1]) \mid \mathbb{E}_{\mu}[X] \leq m \} \]
Reduction theorems

\[ A = \left\{ (f, \mu) \mid f: \chi_1 \times \cdots \times \chi_m \to \mathbb{R}, \mu = \mu_1 \otimes \cdots \otimes \mu_m, \mathcal{G}(f, \mu) \leq 0 \right\} \]

\[ \mathcal{G}(f, \mu) \leq 0 \iff \begin{cases} \text{n'} generalized moment constraints on } \mu, & \mathbb{E}_\mu[\varphi_{f,j}^j] \leq 0 \\ \text{n}_k \text{ generalized moment constraints on } \mu_k, & \mathbb{E}_{\mu_k}[\psi_{f,k,j}^j] \leq 0 \end{cases} \]

Theorem

\[ \sup_{(f, \mu) \in A} \mathbb{E}_\mu[qf] = \sup_{(f, \mu) \in A_\Delta} \mathbb{E}_\mu[qf] \]

\[ A_\Delta = \left\{ (f, \mu) \in A \mid \mu_k \text{ is a sum of at most } \text{n'} + \text{n}_k + 1 \text{ weighted Dirac measures on } \chi_k \right\} \]
Reduction of optimization variables

\[ \{ f : \mathcal{X} \to \mathbb{R}, \mu \in \mathcal{P}(\mathcal{X}) \} \]

\[ \begin{align*}
\{ f : \mathcal{X} \to \mathbb{R}, \mu \in \mathcal{P}(\mathcal{X}) \mid \mu &= \sum_{i=1}^{k} \alpha_k \delta_{x_k} \} \\
\{ f : \{1, 2, \ldots, n\} \to \mathbb{R}, \mu \in \mathcal{P}(\{1, 2, \ldots, n\}) \} \\
\{ \{1, 2, \ldots, q\}, \mu \in \mathcal{P}(\{1, 2, \ldots, n\}) \} 
\end{align*} \]
Non-convex and infinite dimensional optimization problems

Can be considered as a **generalization of classical Chebyshev inequalities**

**History of classical inequalities:** Karlin, Studden (1966, Tchebycheff systems with applications in analysis and statistics)

**Connection between Chebyshev inequalities and optimization theory**

- Mulholland & Rogers (1958, Representation theorems for distribution functions)
- Godwin (1973, Manipulation of voting schemes: a general result)
- Olhin & Pratt (1958, A multivariate Tchebycheff inequality)
- Classical Markov-Krein theorem (Karlin, Studden, 1958)
- Dynkin (1978, Sufficient statistics & extreme points)
- Karr (1983, Extreme points of probability measures with applications)
- Artzner et al (1997, risk measures, value at risk, etc…)
- Betsimas & Popescu (2008, convex optimization approach to inequalities in prob. theo.)
$\mathcal{U}(\mathcal{A}) := \sup_{(g, \mu) \in \mathcal{A}} \mathbb{E}_\mu[q_g]$ 

Our work: Further generalization to
• Independence constraints
• More general domains (Suslin spaces) (non metric, non compact)
• More general classes of functions (measurable) (non continuous, non-bounded)
• More general classes of probability measures
• More general constraints (inequalities, on measures and functions)

Theory of majorization
• Marshall & Olkin (1979, Inequalities: Theory of majorization and its applications)
Inequalities of

• Anderson (1955, the integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities)
• Hoeffding (1956, on the distribution of the number of successes in independent trials)
• Joe (1987, Majorization, randomness and dependence for multivariate distributions)
• Bentkus, Geuze, Van Zuijlen (2006, Optimal Hoeffding like inequalities under a symmetry assumption)
• Pinelis (2007, Exact inequalities for sums of asymmetric random variables with applications.
  2008, On inequalities for sums of bounded random variables)

Our proof rely on

• Winkler (1988, Extreme points of moment sets)
• Follows from an extension of Choquet theory (Phelps 2001, lectures on Choquet's theorem) by Von Weizsacker & Winkler (1979, Integral representation in the set of solutions of a generalized moment problem)
• Kendall (1962, Simplexes & Vector lattices)
Application: Optimal concentration inequality

\[ A_{MD} := \left\{ (f, \mu) \mid f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to \mathbb{R}, \mu \in \mathcal{M}(\mathcal{X}_1) \otimes \cdots \otimes \mathcal{M}(\mathcal{X}_m), \mathbb{E}_\mu[f] \leq 0, \text{Osc}_i(f) \leq D_i \right\} \]

\[ \text{Osc}_i(f) := \sup_{(x_1,\ldots,x_m) \in \mathcal{X}} \sup_{x'_i \in \mathcal{X}_i} (f(\ldots,x_i,\ldots) - f(\ldots,x'_i,\ldots)). \]

\[ \mathcal{U}(A_{MD}) := \sup_{(f,\mu) \in A_{MD}} \mu[f(X) \geq a] \]

McDiarmid inequality \[ \mathcal{U}(A_{MD}) \leq \exp \left( -2 \frac{a^2}{\sum_{i=1}^{m} D_i^2} \right) \]
Reduction of optimization variables

\[ A_C := \left\{ (C, \alpha) \mid \alpha \in \bigotimes_{i=1}^{m} \mathcal{M}(\{0, 1\}), \quad \mathbb{E}_\alpha[h^C] \leq 0 \right\} \]

\[ h^C : \{0, 1\}^m \rightarrow \mathbb{R} \]

\[ t \rightarrow a - \min_{s \in C} \sum_{i : s_i \neq t_i} D_i \]

\[ \mathcal{U}(A_C) := \sup_{(C, \alpha) \in A_C} \alpha[h^C \geq a] \]

Theorem

\[ \mathcal{U}(A_{MD}) = \mathcal{U}(A_C) \]
## Explicit Solution m=2

### Theorem

\[
\mathcal{U}(\mathcal{A}_{MD}) = \begin{cases} 
0 & \text{if } D_1 + D_2 \leq a \\
\frac{(D_1 + D_2 - a)^2}{4D_1 D_2} & \text{if } |D_1 - D_2| \leq a \leq D_1 + D_2 \\
1 - \frac{a}{\max(D_1, D_2)} & \text{if } 0 \leq a \leq |D_1 - D_2| 
\end{cases}
\]

### OUQ bound

\(a = 1\)

\[
C = \{(1, 1)\}
\]

\[
h^C(s) = a - (1 - s_1)D_1 - (1 - s_2)D_2
\]

### Corollary

If \(D_1 \geq a + D_2\), then

\[
\mathcal{U}(\mathcal{A}_{MD})(a, D_1, D_2) = \mathcal{U}(\mathcal{A}_{MD})(a, D_1, 0)
\]
Explicit Solution $m=3$

**Theorem**

$m = 3 \quad D_1 \geq D_2 \geq D_3$

$U(A_{MD}) = \max(F_1, F_2)$

\[ F_1 := \begin{cases} 
0 & \text{if } D_1 + D_2 + D_3 \leq a \\
\frac{(D_1+D_2+D_3-a)^3}{27D_1D_2D_3} & \text{if } D_1 + D_2 - 2D_3 \leq a \leq D_1 + D_2 + D_3 \\
\frac{(D_1+D_2-a)^2}{4D_1D_2} & \text{if } D_1 - D_2 \leq a \leq D_1 + D_2 - 2D_3 \\
1 - \frac{a}{\max(D_1,D_2)} & \text{if } 0 \leq a \leq D_1 - D_2 
\end{cases} \]

\[ F_2 := \max_{i \in \{1,2,3\}} \phi(\gamma_i)\psi(\gamma_i) \]

\[ (1 + \gamma)^3 - \frac{5D_2 - 2D_3}{2D_2 - D_3}(1 + \gamma)^2 + \frac{4D_2 - a}{2D_2 - D_3} = 0, \]
Caltech Small Particle Hypervelocity Impact Range

We want to certify that

\[
P[G = 0] \leq \epsilon
\]
Caltech Hypervelocity Impact Surrogate Model

- **Plate thickness** \( h \in \mathcal{X}_1 := [1.524, 2.667] \text{ mm}, \)
- **Plate Obliquity** \( \alpha \in \mathcal{X}_2 := [0, \frac{\pi}{6}] \),
- **Projectile velocity** \( v \in \mathcal{X}_3 := [2.1, 2.8] \text{ km} \cdot \text{s}^{-1} \).

Thickness, obliquity, velocity: independent random variables

Mean perforation area: in between 5.5 and 7.5 mm\(^2\)

Deterministic surrogate model for the perforation area (in mm\(^2\))

\[
H(h, \alpha, v) = K \left( \frac{h}{D_p} \right)^p (\cos \alpha)^u \left( \tanh \left( \frac{v}{v_{bl}} - 1 \right) \right)^m + \]

\( H_0 = 0.5794 \text{ km} \cdot \text{s}^{-1}, \quad s = 1.4004, \quad n = 0.4482, \quad K = 10.3936 \text{ mm}^2, \)

\( p = 0.4757, \quad u = 1.0275, \quad m = 0.4682. \quad v_{bl} := H_0 \left( \frac{h}{(\cos \alpha)^n} \right)^s \)
Optimal bound on the probability of non perforation

\[ A := \left\{ (f, \mu) \mid \mu = \mu_1 \otimes \mu_2 \otimes \mu_3, \\
5.5 \, mm^2 \leq \mathbb{E}_\mu[f] \leq 7.5 \, mm^2, \\
f = H \right\} \]

\[ \mathcal{U}(A) := \sup_{(f, \mu) \in A} \mu[f(X) = 0] \]

Application of the reduction theorem

The measure of probability can be reduced to the tensorization of 2 Dirac masses on thickness, obliquity and velocity

\[ \mathcal{U}(A)^{\text{num}} = 37.9\% \]
The optimization variables can be reduced to the tensorization of 2 Dirac masses on thickness, obliquity and velocity.

Support Points at iteration 0
Numerical optimization

Support Points at iteration 150
Numerical optimization

Support Points at iteration 200
Velocity and obliquity marginals each collapse to a single Dirac mass. The plate thickness marginal collapses to have support on the extremes of its range.

Probability non-perforation maximized by distribution supported on minimal, not maximal, impact obliquity. Dirac on velocity at a non extreme value.
Important observations

Extremizers are singular

They identify key players i.e. vulnerabilities of the physical system

Extremizers are attractors
Initialization with 3 support points per marginal

Support Points at iteration 0
Initialization with 3 support points per marginal

Support Points at iteration 500
Initialization with 3 support points per marginal

Support Points at iteration 1000
Initialization with 3 support points per marginal

Support Points at iteration 2155
Initialization with 5 support points per marginal

Support Points at iteration 0
Initialization with 5 support points per marginal

Support Points at iteration 1000
Initialization with 5 support points per marginal

Support Points at iteration 3000
Initialization with 5 support points per marginal

Support Points at iteration 7100
Seismic Safety Assessment of Truss Structures
$F(a) = \min_i (S_i - \|Y_i\|_\infty)$

$S_i$: Yield strain of member $i$

$Y_i(t)$: Axial strain of member $i$
$F_{\text{min}}(\text{Yield Strain - Axial Strain})$

We want to certify that

$$\mathbb{P} \left[ F(\alpha) \leq 0 \right] \leq \epsilon$$
Historical Data Method

1940 Elcentro

2010 Haiti

1999 Izmit
Matsuda-Asano shape function (mean power spectrum)

\[ s(\omega) := \frac{\omega^2 \omega^2}{(\omega^2_g - \omega^2)^2 + 4\xi^2 g \omega^2 \omega^2} \]
**OUQ vs Filtered White Noise**

**A**: Set of measures $\mu$ on $A$

- Maximum grounded acceleration bounded
- Mean power spectrum given

Diagram:
- Hexagon labeled $A$
- Star at each corner labeled $\mu$
- Arrow pointing to $\mu$ from the center labeled $\mu wn$
Modeling in the frequency domain

\[ a := \sum_{k=1}^{W} \left( (A_{6k-5}, A_{6k-4}, A_{6k-3}) \cos(2\pi \omega_k t) \right. \]

\[ \quad + \left. (A_{6k-2}, A_{6k-1}, A_{6k}) \sin(2\pi \omega_k t) \right) \]

\[ \omega_k := \frac{k}{T} \quad T = 20 \text{ s} \quad W := 100 \]

\[ \frac{1}{T} \int_0^T |a|^2 \, dt \leq \frac{a_{\text{max}}^2}{2} \]

\[ A := (A_1, \ldots, A_{6W}) \]

\[ \mathbb{P} \left[ A \in B(0, a_{\text{max}}) \right] = 1 \]
Esteva’s semi-empirical expression

\[ a_{\text{max}} := \frac{a_0 e^{\lambda M_L}}{(R_0 + R)^2} \]

*R*: source to site distance
\[ \mathbb{E} \left[ A_i^2 \right] = b_i \]

\[ b_{6k-j} = \frac{a_{\text{max}}^2}{12} \frac{s(\omega_k)}{\sum_{n=1}^{W} s(\omega_n)} \]  \quad j \in \{0, \ldots, 5\}

**Matsuda-Asano shape function**

\[ s(\omega) := \frac{\omega^2 \omega^2}{(\omega_g^2 - \omega^2)^2 + 4\xi_g^2 \omega_g^2 \omega^2} \]

\( \omega_g \): natural frequency of the site.

\( \xi_g \): natural damping factor of the site.
The OUQ problem

**A**: Set of measures $\mu$ on $A$

$\mu[\{A \in B(0, a_{\text{max}})\}] = 1$

$\mathbb{E}_\mu[A_i^2] = b_i$

$$\sup_{\mu \in A} \mu[F(A) \leq 0]$$
Reduction to weighted sums of Diracs

**A:** Set of measures $\mu$ on $A$

$$\mu \left[ A \in B(0, a_{\max}) \right] = 1 \quad \mathbb{E}_\mu [A_i^2] = b_i$$

$$\mu = \sum_{j=1}^{6W+1} p_j \delta_{Z_j}$$

$$Z_j \in \mathbb{R}^{6W} \cap B(0, a_{\max})$$

$$\sum_{j=1}^{6W+1} p_j = 1$$

The reduced problem is of dimension $(6W+1) \times (6W+1)$
Reduction based on strong duality

\[
\sup_{\mu \in \mathcal{A}} \mu \left[ F(A) \leq 0 \right] = \inf_{H \in \mathbb{R}^{6W}} \sup_{x \in \mathbb{R}^{6W} \cap B(0, a_{max})} \chi(x) + \sum_{i=1}^{6W} H_i (b_i - x_i^2)
\]

\[
\chi(x) = 0 \text{ if } A = x \text{ does not fail}
\]

\[
\chi(x) = 1 \text{ if } A = x \text{ fails}
\]

The reduced problem is of dimension: 12W and convex in H
Vulnerability Curves (vs earthquake magnitude)
Modeling in the frequency domain

Number of truss structure (electric tower) members: 198

Number of random Fourier coefficients (with unknown pdf): 600

Dimension of the Reduced Problem: 1200

Reduced problem solved with a Differential Evolution Algorithm modified to use large-scale parallel computing resources

Differential Evolution Algorithm population size: 40

High performance computer cluster: 88 cores
  shc (PSAAP) with 11 core-4 nodes (44 total)
  foxtrot (DANSE) with 4 core-12 nodes, 11/12 (44 total)

Convergence time: 15 hours

Number of iterations: 2000

Number of function evaluations: 35,000 to 50,000
Modeling in the physical domain

\[ a = \psi \star s \]
\[ s(t) := \sum_{i=1}^{B} X_i s_i(t) \quad B = 20 \]

\[ s_i(t) \quad \text{\(\tau_i\) independent} \]
\[ \tau_i \in [0, 30 \text{s}] \]
\[ 1 \text{s} \leq \mathbb{E}[\tau_i] \leq 2 \text{s} \]

\[ X_i \in [-a_{\text{max}}, a_{\text{max}}]^3 \]
\[ X_{i,j} \text{ independent} \]
\[ \mathbb{E}[X_i] = 0 \]
$a = \psi \star s$
Application of OUQ reduction theorems

\[ \mathcal{U}(A) := \sup_{(F, \mu) \in A} \mu [F \leq 0] \]

The optimum can be achieved by

- Handling \( \psi \) as a deterministic optimization variable.
- Assuming that the measure on each \( X_{i,j} \) is the tensorization of two Dirac masses in \([ -a_{\text{max}}, a_{\text{max}} ]\).
- Assuming that the measure on each \( \tau_i \) is the weighted sum of 2 Dirac masses in \([ 0, \tau_{\text{max}} ]\).

Identification of the weakest elements
Power Spectrum
Positions (abscissa, in m · s\(^{-2}\)) and weights (ordinates) of the Dirac masses associated with the measure of probability on X\(_1\), . . . , X\(_B\) at the extremum for earthquakes of magnitude ML = 6, ML = 6.5 and ML = 7. The positions in abscissa correspond to the possible amplitudes of the impulses Xi.
Second reduction (positions of the Diracs)

\[ \mathcal{A} = \{(f, \mu) \in \mathcal{G} \times \bigotimes_{i=1}^{m} \mathcal{M}(\chi_i) \mid \Phi(f, \mu) \leq 0\} \]

\( \mathcal{G} \subset \mathcal{F} \): Set of real measurable functions on \( \chi := \chi_1 \times \cdots \times \chi_m \)

\( \Phi(f, \mu) \leq 0 \iff \mathbb{E}_\mu[\varphi_j \circ f] \leq 0 \quad 1 \leq j \leq n \)

**Theorem**

\[ \sup_{(f, \mu) \in \mathcal{A}} \mathbb{E}_\mu[q \circ f] = \sup_{(h, \alpha) \in \mathcal{A}_\mathcal{D}} \mathbb{E}_\alpha[q \circ h] \]

\[ \mathcal{A}_\mathcal{D} = \{(h, \alpha) \in \mathcal{G}_\mathcal{D} \times \bigotimes_{i=1}^{m} \mathcal{M}(\mathcal{D}') \mid \Phi(f, \mu) \leq 0\} \]

\[ \mathcal{D}' = \{0, \ldots, n\} \]

\( \mathcal{G}_\mathcal{D} \subset \mathcal{F}_\mathcal{D} \): Real functions on \( \mathcal{D} := \{0, \ldots, n\}^m \)
\[ \mathcal{G}_D := \mathcal{F} \left[ \mathcal{G} \times \bigotimes_{i=1}^{m} \Delta_n(x_i) \right] \]

\[ \mathcal{F}: \mathcal{F} \times \bigotimes_{i=1}^{m} \Delta_n(x_i) \longrightarrow \mathcal{F}_D \]

\[ (f, \bigotimes_{i=1}^{m} \left( \sum_{k=0}^{n} \alpha_i^k \delta_{x_i^k} \right)) \longrightarrow (s_1, \ldots, s_m) \rightarrow f(x_1^{s_1}, \ldots, x_m^{s_m}) \]

\[ f: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{R} \]

\[ h: \{0, 1\} \times \{0, 1\} \rightarrow \mathbb{R} \]
Reduction of optimization variables

\[ \{ f : \mathcal{X} \rightarrow \mathbb{R}, \mu \in \mathcal{P}(\mathcal{X}) \} \]

\[ \{ f : \mathcal{X} \rightarrow \mathbb{R}, \mu \in \mathcal{P}(\mathcal{X}) | \mu = \sum_{i=1}^{k} \alpha_k \delta_{x_k} \} \]

\[ \{ f : \{1, 2, \ldots, n\} \rightarrow \mathbb{R}, \mu \in \mathcal{P}(\{1, 2, \ldots, n\}) \} \]

\[ \{ \{1, 2, \ldots, q\}, \mu \in \mathcal{P}(\{1, 2, \ldots, n\}) \} \]