An HIV model: Theoretical Analysis and Experimental Verification

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Contemporary Approaches in Mathematical Epidemiology, Ecology and Population Dynamics

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- Jun-Yuan Yang
- Xiao-Yan Wang
- Feng-Qin Zhang
Outline

1 Introduction
2 Model
3 Equilibria
4 Parameter Estimation
5 Conclusion
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1 Introduction

2 Model

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5 Conclusion
Introduction

Model

Equilibria

Parameter Estimation

Conclusion
Spread of HIV in China

Figure 7

Prevalence Among Injecting Drug Users

- 0 - 1.0%
- 1.1 - 3.0%
- 3.1 - 11.0%
- 35.0 - 85.1%
- No data

Source: UNAIDS
Growth of HIV in China

Introduction

Model

Equilibria

Parameter Estimation

Conclusion

![Graph showing the growth of HIV in China from 1997 to 2007](image)

- Injection-drug use
- High-risk heterosexual contact
- Male-to-male sexual contact
- Blood donation/receipt
- Other

No. of cases vs Year of diagnosis
Leading cause of existence of HIV in China

- Visit Doctors

- Floating population
Leading cause of existence of HIV in China

Visit Doctors

Floating population
HIV Model in China

Susceptible (S)

Infected (I)
Infected undetected (I_u)
Model and parameters

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \frac{[(\beta_k + \beta_{uk})I + (\beta_{ku} + \beta_u)I_u]S}{N} - \mu S, \\
\frac{dI}{dt} &= \frac{(\beta_k I + \beta_{ku}I_u)S}{N} - (\mu + \alpha)I, \\
\frac{dI_u}{dt} &= \frac{(\beta_{uk} I + \beta_u I_u)S}{N} - (\mu + \alpha)I_u.
\end{align*}
\]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Cite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
<td>Input rate from outside</td>
<td>Value</td>
</tr>
<tr>
<td>$\beta_k, \beta_{uk}$</td>
<td>Detected transmission coefficient</td>
<td>Estimated</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Nature death rate</td>
<td>0.01</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Death rate due to HIV</td>
<td>0.2</td>
</tr>
<tr>
<td>$\beta_u, \beta_{ku}$</td>
<td>Undetected transmission coefficient</td>
<td>Estimated</td>
</tr>
</tbody>
</table>
Investigation of Model equilibria

\[
\begin{align*}
0 &= \Lambda - \frac{[\beta_k I + \beta_{uk} I_u]S}{N} - \mu S, \\
0 &= \frac{(\beta_k I + \beta_{ku} I_u)S}{N} - (\mu + \alpha)I, \\
0 &= \frac{(\beta_{uk} I + \beta_u I_u)S}{N} - (\mu + \alpha)I_u.
\end{align*}
\]
Model Equilibria

Disease Free Equilibrium:

\[ E_0 = \left( \frac{\Lambda}{\mu}, 0, 0 \right) \]

Endemic Equilibrium:

\[ E^* = (S^*, I^*, I_u^*) \]
\( \beta_{uk} = 0 \),

Boundary equilibria \( E_1 = (N_1, I_1, 0) \),

Co-existence equilibrium \( E^* = (N^*, I^*, I^*_u) \), where

\[
N_1 = \frac{\Lambda}{\beta_k - \alpha}, \quad I_1 = \frac{\Lambda(\beta_k - \mu - \alpha)}{(\mu + \alpha)(\beta_k - \alpha)}.
\]

\[
N^* = \frac{\Lambda}{\beta_u - \alpha}, \quad I^* = \frac{\beta_{ku} \Lambda(\beta_u - \mu - \alpha)}{(\beta_u - \alpha)(\mu + \alpha)(\beta_k - \beta_{ku} - \beta_u)}.
\]
The next generation approach

We define

\[ X = (I, I_u, S). \]

\[ F = \left( \frac{(\beta_k I + \beta_k I_u)S}{N}, \frac{(\beta_{uk} I + \beta_u I_u)S}{N}, 0 \right)^T, \]

\[ V = ((\mu+\alpha)I, (\mu+\alpha)I_u, -\Lambda + \frac{[(\beta_k + \beta_{uk})I - (\beta_k + \beta_u)I_u]S}{N} + \mu S. \]
Calculated $F = D\mathcal{F}(E_0)$ and $V = D\mathcal{V}(E_0)$ as follows

$$F = \begin{pmatrix} \beta_k & \beta_{ku} \\ \beta_{uk} & \beta_u \end{pmatrix}, \quad V = \begin{pmatrix} \mu + \alpha & 0 \\ 0 & \mu + \alpha \end{pmatrix}. $$

Basic reproduction number is spectral radius of $FV^{-1}$

$$R_0 = \frac{\beta_k + \beta_u + \sqrt{(\beta_k + \beta_u)^2 + 4\beta_{ku}\beta_{uk}}}{2(\mu + \alpha)} \frac{1}{2}.$$
Stability of Disease free Equilibrium

**Theorem 1** If $R_0 < 1$, disease free equilibrium $E_0$ is locally asymptotically stable.

**Theorem 2** If $\frac{\beta_k + \beta_{uk}}{\mu + \alpha} < 1$, and $\frac{\beta_u + \beta_{ku}}{\mu + \alpha} < 1$, disease free equilibrium $E_0$ is globally asymptotically stable.
Proof of Theorem 2.

Lyapunov function $V = I + I_u$. Then

\[
\dot{V} = \dot{I} + \dot{I_u} \\
= (\beta_k I + \beta_{ku} I_u)S - (\mu + \alpha)I \\
+ \frac{(\beta_{uk} I + \beta_u I_u)S}{N} - (\mu + \alpha)I_u \\
\leq [(\beta_k + \beta_{uk}) - (\mu + \alpha)]I(t) + [(\beta_u + \beta_{ku}) - (\mu + \alpha)]I_u \\
= (\mu + \alpha)[\frac{\beta_k + \beta_{uk}}{\mu + \alpha} - 1]I(t) + (\mu + \alpha)[\frac{\beta_u + \beta_{ku}}{\mu + \alpha} - 1]I_u
\]
Proof continued

\[
\frac{\beta_k + \beta_{uk}}{\mu + \alpha} < 1, \quad \frac{\beta_u + \beta_{ku}}{\mu + \alpha} < 1 \Rightarrow \dot{V} \leq 0.
\]

\[
\dot{V} = 0 \iff I, I_u = 0
\]

the largest invariant set is \( \left( \frac{A}{\mu}, 0, 0 \right) \) when \( (I, I_u) = (0, 0) \). the set \( \{ x \mid \dot{V} = 0 \} \) is the singleton given by \( E_0 \). Global stability follows by by Lassale’s invariance principle.
Stability of Endemic equilibria

**Theorem** If $\beta_{uk} = 0$, and $\frac{\beta_k}{\mu + \alpha} > 1$, then there exist unique positive boundary equilibrium $E_1$. If $\frac{\beta_u}{\mu + \alpha} > 1$, and $\beta_k > \beta_{ku} + \beta_u$, then there exist unique positive co-existence equilibrium $E^*$. 
Parameter Estimation

Parameters to estimate

\[ x = [\Lambda, \beta_k, \beta_{ku}, \beta_{uk}, \beta_u]^T. \]

Method

Least Square approach :

\[ Min f(x) = \frac{1}{2} \sum_{i=0}^{N_{max}} \delta_i (I_i - \hat{I}_i)^2. \]
### Estimated Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\Lambda$</th>
<th>$\beta_k$</th>
<th>$\beta_{ku}$</th>
<th>$\beta_{uk}$</th>
<th>$\beta_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.0178</td>
<td>0.1337</td>
<td>0.0666</td>
<td>0.6010</td>
<td>0.0215</td>
</tr>
<tr>
<td>Parameters</td>
<td>t-distribution fitting</td>
<td>Exponential fitting</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>------------</td>
<td>------------------------</td>
<td>---------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>(-6887 6888)</td>
<td>(-2419 2419)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_k )</td>
<td>(-726.1 726.4)</td>
<td>(-254.9 255.2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_{ku} )</td>
<td>(-717.5 718.7)</td>
<td>(-251.6 252.8)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_{uk} )</td>
<td>(-663.9 665.7)</td>
<td>(-232.6 234.3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_u )</td>
<td>(-507.2 507.3)</td>
<td>(-178.1 178.2)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• Observed that $R_0 = 3.8221$ which means HIV is endemic in the city.

• Transmission rates to the undetected compartment should be controlled in the city.

• The government needs to adopt more sophisticated strategies to control the spread of this disease in the undetected compartment.