Optimal Asset Liquidation using Limit Order Book Information

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Monday July 9, 2012
SIAM Conference on Financial Mathematics & Engineering
Minneapolis, MN
Optimal Liquidation

How to liquidate $X$ shares of an asset?

1. **Macroscopic** time scale:
   - Horizon $\bar{T} > 0$ over which the shares $X$ need to be liquidated.
   - Depends on \textit{long term} variables: average daily volume, strategic considerations, news events, ...

2. **Mesoscopic** time scale:
   - Trade schedule $0 \leq t_0 \leq t_1 \ldots \leq t_i \leq \ldots \leq t_n = \bar{T}$ for the “child” trades.
   - Depends on \textit{medium term} variables: volatility of the stock, risk aversion of the trader, price impact considerations, ...

3. **Microscopic** time scale:
   - Within a time interval $(t_i, t_{i+1}]$, what is the \textit{timing} and the \textit{type of order} used to liquidate the “child” trade?
   - Depends on \textit{short term} variables: limit order book information.
**Literature: Mesoscopic Time Scale**

1. **Almgren and Chriss (1998)**
   - Objective: maximize risk-adjusted revenues
     \[
     E(R_T^x) - \lambda V(R_T^x).
     \]
   - $R_T^x$ is the revenue from liquidation and $x = (x_0, x_1, ..., x_T)$ is a deterministic trade schedule.
   - Solution: $x_t = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} X$, for $t \in \{0, \ldots, T\}$, and $\kappa$ depends on price volatility, risk aversion, and price impact.

2. **Schied, Schoneborn and Tehranchi (2010)**
   \[
   \sup_{X \in \chi} E[u(R_T^X)]
   \]
   - $X$ is the optimal control within a class $\chi$ of stochastic controls.
   - If the utility function is exponential, $X$ is a deterministic function of time.
The trade schedule
Microscopic Time Scale

- We assume that the trade schedule is given.
- The goal is then to liquidate one lot (the shares $x_t$) in the time window $(t_i, t_{i+1}]$, i.e., what is the optimal time $\tau$ in $[0, T]$ to sell the lot, where $T = t_{i+1} - t_i > 0$.
- $T$ is typically short, e.g., 1 minute.
- For such short time periods, observing the limit order book can be very advantageous in identifying good liquidation times.
- However, latency in the trade execution can diminish this advantage!
Latency

- Latency arises in every trade execution:
  1. Time of datafeed to travel from exchange to execution machine;
  2. The algorithm making a decision;
  3. The order being sent back to the market.

- Latency has no effect on deterministic trade schedules.

- In our model the algorithm will take into account that if a market order is sent at time $t$ it will actually be executed at the best price available at time $t + l$, for latency $l > 0$.

- This worsen the performance of our optimal liquidation algorithm, thus allowing us to quantify the cost of latency.
Outline

1. Optimal liquidation:
   - The efficient price process.
   - Optimal stopping problem.
   - The trade and no-trade regions.

2. Trading with latency.

3. Dynamic programming approximation.

4. Backtesting strategy on TAQ data.

5. Conclusions and future research.
The Efficient Price Process

- The **efficient** or “true” price process

\[ S(t) = S^b(t) + \theta(t), \]

where \( S^b(t) \) is the bid price and \( \theta(t) \) is the **imbalance process**:

\[ \theta(t) = g \left( \frac{B(t)}{A(t) + B(t)} \right), \]

\( A(t) \) is the bid size, \( B(t) \) is the ask size and \( g(\cdot) \) is a cubic polynomial.

- **Assumptions:**
  - \( S(t) \) is a Lévy process,
  - \( S(t) \) is a martingale,
  - \( S^b(t) \) is in \( \mathbb{Z} \) and \( \theta(t) \in [0, 1) \).
The Optimal Liquidation Problem

- Efficient price process: \( S(t) = S^b(t) + \theta(t) \).
- Submitting a sell order at time \( t \) yields payoff
  \[
  S^b(t) = \lfloor S(t) \rfloor \leq S(t).
  \]
- **Goal**: Identify an optimal time \( \tau \) in \([0, T]\) to sell the share and in turn to receive \( \lfloor S(t) \rfloor \), i.e.,
  \[
  V(t, s) = \sup_{t \leq \tau \leq T} E[\lfloor S(\tau) \rfloor | S(t) = s],
  \]
  for \( s \in \mathbb{R} \) and \( t \in [0, T] \), and \( \tau \in \mathcal{T} \), where \( \mathcal{T} \) is the set of stopping times with respect to \( \sigma(S(t))_{t \geq 0} \).
Trade/No-trade Regions

- “Trade” and “No-trade” region

$$D = \{(t, s) \in [0, T] \times \mathbb{R} : V(t, s) = \lfloor s \rfloor\},$$

$$C = \{(t, s) \in [0, T] \times \mathbb{R} : V(t, s) > \lfloor s \rfloor\}.$$

- Liquidation time

$$\tau_D = \inf \{t \geq 0 | S(t) \in D\}.$$

Proposition

$$\tau_D \in \mathcal{T} \text{ and } V(t, s) = \mathbb{E}[\lfloor S(\tau_D) \rfloor | S(t) = s].$$
Structural Properties of Value Function

Proposition
The function $V(t, s)$ satisfies the following properties:

(a) fix $t \in [0, T]$, then $V(t, s)$ is non-decreasing in $s$;
(b) fix $s \in \mathbb{R}$, then $V(t, s)$ is non-increasing in $t$;
(c) $V(t, s + z) = V(t, s) + z$ for all $s \in \mathbb{R}$, $t \in [0, T]$ and $z \in \mathbb{Z}$;
(d) $V(t, z) = z$ for all $t \in [0, T]$ and $z \in \mathbb{Z}$;
(e) $V(T, s) = \lfloor s \rfloor$ for all $s \in \mathbb{R}$. 
State Space Reduction

1. Property (c) shows that

\[ V(s, t) = \sup_{t \leq \tau \leq T} \mathbb{E}[\lfloor S(\tau) \rfloor | S(t) = s] = \sup_{t \leq \tau^\theta \leq T} \mathbb{E}[\lfloor S(\tau^\theta) \rfloor | S(t) = s], \]

where \( \tau^\theta \) are stopping times adapted to \((\theta(t))_{t \geq 0}\).

2. Further, for \( \tau \in \mathcal{T} \):

\[
\mathbb{E}[\lfloor S(\tau) \rfloor | S(t) = s] = \mathbb{E}[S(\tau) - \theta(\tau) | S(t) = s] \\
= s - \mathbb{E}[\theta(\tau) | S(t) = s].
\]

Hence, the problem \( V(t, s) \) is equivalent to

\[ V^\theta(t, u) = \inf_{t \leq \tau^\theta \leq T} \mathbb{E}[\theta(\tau^\theta) | \theta(0) = u], \]

and \( V(s, t) = s - V^\theta(t, s - \lfloor s \rfloor) \).
Optimal Liquidation based on Minimizing Imbalance

Define

\[ D^\theta = \left\{ (t, u) \in [0, T] \times [0, 1) : V^\theta(t, u) = u \right\}, \]

\[ C^\theta = \left\{ (t, u) \in [0, T] \times [0, 1) : V^\theta(t, u) < u \right\}. \]

**Proposition**

There exists a non-decreasing function \( w^* : [0, T] \rightarrow [0, 1] \) with \( w^*(T) = 1 \), such that \( D^\theta = \{(u, t) \in [0, 1) \times [0, T] : u \leq w^*(t)\} \).
Trade/no Trade Regions

Jump Process, $\lambda(T) = 500$, $K = 0.4$, $\sigma = 0.005$

Latency = 0T

Timesteps: 10000
States: 500
Sensitivity of the Trade Region

As the volatility of the price process $S(t)$ increases one can liquidate less aggressively (assuming no risk aversion).
Trading with Latency

- A trade triggered at time $t$ is executed at time $t + l$ for $l > 0$.

- Consider

$$V^l(t, s) = \sup_{t \leq \tau \leq T-l} \mathbb{E}[S^b(\tau + l)|S(t) = s],$$

where $\tau \in \mathcal{T}$.

- Define the payoff function $G^l(s) = \mathbb{E}[S^b(l)|S(0) = s]$ for $s \in \mathbb{R}$, then, for $\tau \in \mathcal{T}$,

$$\mathbb{E}[S^b(\tau + l)|S(t) = s] = \mathbb{E}[\mathbb{E}[S^b(\tau + l)|S(\tau)]|S(t) = s]$$

$$= \mathbb{E}[G^l(S(\tau))|S(t) = s].$$

- Therefore

$$V^l(t, s) = \sup_{t \leq \tau \leq T-l} \mathbb{E}[G^l(S(\tau))|S(0) = s].$$
Latency is Costly

**Proposition**

Fix $t \in [0, T], s \in \mathbb{R}$, then $V'(t, s)$ is non-increasing in $l$ for $l \in [0, T]$. 
Trade/No-Trade Regions with Latency

The “trade region” is still connected, but the “no-trade” region does not need to be connected anymore:

Proposition

There exists a non-decreasing function \( w_i^* : [0, T] \rightarrow [0, 1] \) and a non-increasing function \( v_i^* : [0, T] \rightarrow [0, 1] \), with \( v_i^* \leq w_i^* \), \( w_i^*(t) = 1 \) for \( t \in [T - l, T] \) and \( v_i^* = 0 \) for \( t \in [T - l, T] \), such that

\[
D^{\theta, l} = \{(t, u) \in [0, T] \times [0, 1) : v_i^*(t) \leq u \leq w_i^*(t)\}.
\]
Trade/No-Trade Regions with Latency cont.

The red line shows the optimal policy without adjustment.
Dynamic Program

- Bellman’s recursion:

\[ V_{E,N}^{\theta,l}(n, i) = \max \left\{ \tilde{G}^{\theta,l}(i), \mathbb{E}[V_{E,N}^{\theta,l}(n + 1, \tilde{\theta}(n + 1)) | \tilde{\theta}(n) = i] \right\}, \]

for \( i \in \{0, \ldots, N\} \) and \( n \in \{0, \ldots, N\} \).

- Conditional probability:

\[ \mathbb{E}[V_{E,N}^{\theta,l}(\tilde{\theta}(n + 1), n + 1) | \tilde{\theta}(n) = i] = \sum_{k=1}^{E} p_{ik} V_{E,N}^{\theta,l}(n + 1, k). \]
Discretization Convergence

As $N \to \infty$ and $E \to \infty$ the boundary between trade and no-trade region converges to a smooth curve.
Backtesting

- Backtesting on TAQ data for 5-years US treasury bonds for 21 days (July 2010).
- Assume that one lot is traded per minute (avoids trading on the same quote).
- Need a price model for

\[ S(t) = S^b(t) + \theta(t), \]

the bid price \( S^b(t) \) can be observed in TAQ data.
- Many possibilities to construct \( \theta(t) \) based on limit order book data.
Imbalance Process

- We use
  \[ \theta(t) = g \left( \frac{B(t)}{A(t) + B(t)} \right) , \]
  where \( A(t) \) (\( B(t) \)) is best ask (bid) size and \( g(\cdot) \) is a cubic polynomial with constraints \( g(0) = 0, g(0.5) = 0.5 \) and \( g(1) = 1 \) (leaving 1 degree of freedom).
- This transformation makes the stationary distribution almost uniform, which is necessary since \( S(t) \) is a Lévy process.
Empirical Evidence for Price Model

Empirical observed imbalance $\theta(t)$ **conditioned a trade occurs on the next quote** gives credential that traders use similar trading strategies as described.

![Histogram of \( \theta_n \) before trade (trade size = 1)](image)
Optimal Stopping vs. TWAP Strategy

- The time-weighted average price (TWAP) strategy liquidates one share per minute independently of the state of the limit order book. (Only trades when spread=1).

- Consider residuals $R = S_{\tau}^b - S_0^b$, where $\tau$ is the stopping time from the optimal stopping problem $V(t,s)$ calibrated to a pure jump process $S(t)$.

- Compare 5,649 intervals of length 1 minute.

- **Without latency** the optimal liquidation strategy saves on average 26 $ per share, i.e., 1/3 of the spread (Spread is 78$ for 5 yrs US-treasury bonds):

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{E}[R]$</th>
<th>$\sigma(\hat{R})$</th>
<th>$P$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal policy vs. TWAP</td>
<td>26.26 $</td>
<td>49.14 $</td>
<td>$2.64 \cdot 10^{-200}$</td>
</tr>
</tbody>
</table>
Cost of Latency

- **Cost of latency:**

\[
COL = \mathbb{E}[S_b(\tau^0 + l) - S_b(\tau^0)],
\]

where \(\tau^0\) is the stopping time induced by \(V(t, s)\).

- **Adjusted cost of latency:**

\[
COL_{adj} = \mathbb{E}[S_b(\tau^l + l) - S_b(\tau^0)],
\]

where \(\tau^l\) is the stopping time induced by the adjusted problem \(V^l(t, s)\).

- Note, we do not calculate the COL with respect to the TWAP strategy, but with respect to the **optimal strategy with no latency**.
The Cost of Latency cont.

- 10ms latency $\approx 10\$ per share.
- For latencies $\geq 2000\text{ms}$ (i.e., 2 secs) the advantage of observing the limit order book diminishes (performance becomes similar to TWAP).
- Adjusting the liquidation policy brings only minor improvement in the performance.
Conclusions

- We consider an optimal stopping problem that depends on:
  - Information found in the order book;
  - Latency;
  - The time left to catch up with the TWAP algorithm.
- The solution comes in the form of a trade/no-trade regions in the imbalance process.
- We estimate model parameters with ”Level II” trades and quotes data.
- We find that our optimal liquidation algorithm significantly outperforms a TWAP algorithm.
- We quantify the cost of latency.
- **Future research:**
  1. Modeling limit order executions.
  2. Models of $S(t)$ for which the execution regions have a closed form.
THANK YOU!
If you wish to have a copy of the paper, please send me an email at sashastoiko@gmail.com