Adaptive mean-variance optimal order execution and Dawson–Watanabe superprocesses

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SIAM Conference on Financial Mathematics and Engineering 2012
Minneapolis
July 9, 2012
The Almgren–Chriss market impact model


\[ x(t) = \text{asset position of investor at time } t \]
assumed to be an absolutely continuous function of time

realized price process:

\[ S_t^x = S_t^0 + \gamma \int_0^t \dot{x}(s) ds + \eta \dot{x}(t) \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

unaffected permanent temporary
price process impact impact
Optimization over adapted and absolutely continuous trajectories \((x(t))_{0 \leq t \leq T}\) with \(x_0\) given and liquidation time constraint \(x(T) = 0\)

The costs of such a strategy are given by

\[
\text{costs}(x) = x_0 S_0 - \int_0^T S_t^x (-\dot{x}(t)) \, dt
\]

\[
= \frac{\gamma}{2} x_0^2 - \int_0^T x(t) \, dS_t^0 + \eta \int_0^T \dot{x}(t)^2 \, dt
\]
Practitioners like to find adaptive strategies that minimize a mean-variance functional

$$\mathbb{E}[\text{costs}(x)] + \lambda \text{var}(\text{costs}(x))$$

But this is not so easy due to the time inconsistency of the variance:

$$\text{var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$$

(see e.g. Lorenz & Almgren (2011)).

Continuous re-optimization leads to the alternative time-consistent cost function

$$\mathbb{E}\left[ \eta \int_0^T |\dot{x}(t)|^2 \, dt + \lambda \int_{[0,T]} |x(t)|^2 \, d[S^0]_t \right]$$

This was observed, e.g., by Konishi & Makimoto (2001), Schöneborn (2008); see also Björk et al. (2010), Ekeland & Lazrak (2008), Forsyth et al. (2011)
Almost all price processes $S^0$ in financial mathematics are of the form

$$S^0_t = \phi(Z_t)$$

For a (time-inhomogeneous) Markov process $Z = (Z_t, P_{r,z})$.

**Generalized finite-fuel control problem:** minimize

$$E_{0,z} \left[ \int_0^T |\dot{x}(s)|^p \eta(Z_s) \, ds + \int_{[0,T]} |x(s)|^p A(ds) \right]$$

over progressively measurable and absolutely continuous strategies $x$ with $x(0) = x_0$ and $x(T) = 0$

Here:

- $A$ is a nonnegative additive functional of $Z$. E.g., $A(dt) = d[\phi(Z)]_t = d[S^0]_t$
- $\eta$ is a strictly positive function

W.l.o.g.: $x$ is monotone
**Heuristic solution**

Assume $A(du) = \alpha(Z_u) du$ and let

$$V(t, z, x_0) := \inf_{x(\cdot)} E_{t,z} \left[ \int_{t}^{T} |\dot{x}(u)|^p \eta(Z_u) \, du + \int_{t}^{T} |x(u)|^p a(Z_u) \, du \right].$$

Optimal control suggest that $V$ should satisfy the following HJB equation

$$\frac{\partial V}{\partial t}(t, z, x_0) + \inf_{\xi} \left\{ \eta(z)|\xi|^p + \frac{\partial V}{\partial x_0}(t, z, x_0)\xi \right\} + \alpha(z)|x_0|^p + L_tV(t, z, x_0) = 0$$

with singular terminal condition

$$V(T, z, x_0) = \begin{cases} 0 & \text{if } x_0 = 0, \\ +\infty & \text{otherwise}. \end{cases}$$
We make the ansatz
\[ V(t, z, x_0) = |x_0|^p v(t, z) \]
for some function \( v \). Plugging this ansatz into the HJB eqn, minimizing over \( \xi \), and dividing by \( |x_0|^p \), yields the PDE
\[
v_t - \frac{1}{\beta \eta^\beta} v^{1+\beta} + \alpha + L_t v = 0, \quad v(T, z) = +\infty,
\]
where
\[ \beta = \frac{1}{p - 1}. \]
Moreover, the minimizing \( \xi \) in the HJB eqn is given by \( \xi = -x v^\beta / \eta^\beta \), which suggests that the optimal strategy is a solution of the o.d.e.
\[
\dot{x}(u) = -\frac{x(u) v(u, Z_u)^\beta}{\eta(Z_u)^\beta},
\]
i.e.,
\[
x(u) = x_0 \exp \left( \int_t^u - \frac{v(s, Z_s)^\beta}{\eta(Z_s)^\beta} \, ds \right).
\]
Dynkin (1991, 1992): for uniformly elliptic diffusions on $\mathbb{R}^d$, constant $\gamma$, and $\beta \in (0, 1]$ ( $\iff p \geq 2$), singular PDEs of the form

$$v_t - \gamma v^{1+\beta} + \alpha + L_t v = 0,$$

$$v(T, z) = +\infty,$$

can be solved by means of **Dawson–Watanabe superprocesses** (and hence via Monte Carlo techniques ...)

Dawson–Watanabe superprocesses are measure-valued Markov processes motivated by describing the spatial evolution of colonies of bacteria ...
\[ X_t = \sum \delta Z_t \]
The superprocess with $\beta = 1$ is obtained by starting with individual $N$ particles, speeding up branching by a factor $N$, assigning mass $\frac{1}{N}$ to each particle, and then sending $N$ to infinity.

$$X_t = \sum \delta Z_t^i$$
The superprocess $X = (X_t, \mathbb{P}_{r,\mu})$ can be characterized by its Laplace functionals:

$$
\mathbb{E}_{r,\mu}[e^{-\langle f, X_T \rangle}] = e^{-\langle v(r, \cdot), \mu \rangle}, \quad f \geq 0 \text{ and bounded},
$$

where $\langle f, \mu \rangle$ is shorthand for $\int f \, d\mu$ and $v$ is the unique positive mild solution of the quasilinear PDE

$$
v_t + L_t v - \gamma v^{1+\beta} = 0
$$

$$
v(T, z) = f(z)
$$

That is

$$
v(r, z) = E_{r,z}[f(Z_T)] - E_{r,z} \left[ \gamma \int_r^T v(t, Z_t)^{1+\beta} \, dt \right]
$$

Remark: Note that the superprocess is an affine process
**Example:** Taking $f$ equal to a constant $k > 0$, we see that

$$v(r, z) = k - E_{r,z} \left[ \gamma \int_r^T v(t, Z_t)^{1+\beta} \, dt \right]$$

is solved by

$$v_k(r, z) = \frac{1}{(k^{-\beta} + \gamma \beta (T - t))^{1/\beta}}$$

Sending $k$ to infinity yields

$$- \log \mathbb{P}_{r,\delta_z} [X_T = 0] = - \lim_{k \uparrow \infty} \log \mathbb{E}_{r,\delta_z} [e^{-k\langle 1, X_T \rangle}]$$

$$= \lim_{k \uparrow \infty} v_k(r, z)$$

$$= \frac{1}{(\gamma \beta (T - r))^{1/\beta}}.$$
The characterization of Laplace functionals extends to nonnegative additive functionals $A$ of $Z$ of the form

$$A[r, t] = \sum_{r \leq t_i \leq t} f_i(Z_{t_i})$$

by means of the $J$-functional

$$I_A = \sum_i \langle f_i, X_{t_i} \rangle$$

and to suitable limits thereof

$$-\log \mathbb{E}_{r, \mu}[e^{-I_A}] = \langle v(r, \cdot), \mu \rangle,$$

where $v(r, z)$ solves

$$v(r, z) = E_{r, z}[A[r, T]] - \gamma E_{r, z} \left[ \int_r^T v(s, Z_s)^{1+\beta} \, ds \right], \quad 0 \leq r \leq T.$$
**Problem:** For \( S_t = \phi(Z_t) \), the additive functional \( A(dt) = d[S]_t \) may not be of this form unless \( Z \) is a path process, e.g.,

\[
Z_t = (S_{s\wedge t})_{s \geq 0}
\]

In this case, \( A(dt) = d[S]_t \) can be approximated by additive functionals

\[
A_n[r, t] = \sum_{r \leq t_{i-1}, t_i \leq t} \left( Z_{t_i}(t_i) - Z_{t_i}(t_{i-1}) \right)^2 = f_i(Z_{t_i})
\]

The corresponding superprocess is the **historical superprocess** (Dawson & Perkins (1991), Dynkin (1991))

But in this situation there is no hope to get smoothness of the solutions of the corresponding log–Laplace equations \( \sim \) we will work only with mild solutions here
Theorem 1 (Case $\eta = 1$). For $p \geq 2$ let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and take $A$ such that $E_{r, z}[A[r, T]^q] < \infty$ for all $r, z$. Let $I_A$ be the corresponding $J$-functional of the superprocess with parameters $Z$, $\gamma := p - 1$, and $\beta := \frac{1}{\gamma}$, and define the function

$$v_\infty(r, z) = -\log E_{r, \delta_x} [e^{-I_A} \cdot 1_{\{X_T = 0\}}].$$

For

$$\phi(\xi, \eta) := \xi^p - p\eta^{p-1}\xi + (p - 1)\eta^p, \quad \xi, \eta \geq 0$$

the costs of any admissible strategy $x$ are

$$E_{0, z} \left[ \int_0^T |\dot{x}(s)|^p \, ds + \int_{[0, T]} |x(s)|^p A(ds) \right]$$

$$= |x_0|^p v_\infty(0, z) + E_{0, z} \left[ \int_0^T \phi(|\dot{x}(t)|, |x(t)|v_\infty(t, Z_t)^{1/(p-1)}) \, dt \right]$$

The unique strategy minimizing these costs is

$$x^*(t) := x_0 \exp \left( -\int_0^t v_\infty(s, Z_s)^\beta \, ds \right)$$

and the minimal costs are $|x_0|^p v_\infty(0, z)$.
Idea of proof:

First step: probabilistic verification argument without smoothness

For $k \in \mathbb{N}$,

\[
(1) \quad v_k(r, z) = -\log \mathbb{E}_{r, \delta z} [e^{-I_{A_1} - k < 1, X_T}].
\]

Then $v_k \to v_\infty$ and $v_k$ uniquely solves

\[
v_k(r, z) = k + E_{r, z}[A[r, T]] - \gamma E_{r, z} \left[ \int_r^T v_k(s, Z_s)^{1+\beta} \, ds \right].
\]

For an admissible strategy $x \geq 0$, let

\[
(2) \quad C^k_t := \int_0^t |\dot{x}(s)|^p \, ds + \int_{[0, t]} x(s)^p A(ds) + x(t)^p v_k(t, Z_t).
\]

Show then

\[
(3) \quad C_t \longrightarrow C_T := \int_0^T |\dot{x}(s)|^p \, ds + \int_{[0, T]} x(s)^p A(ds) \quad P_{0, z}-a.s. \text{ as } t \uparrow T.
\]
We next let

\[ M_t^k := k + E_{0,z} \left[ A[0, T] - \gamma \int_0^T v_k(s, Z_s)^{1+\beta} \, ds \bigg| \mathcal{F}_t \right] \]

so that

\[ V_t := v_k(t, Z_t) = M_t^k - A[0, t] + \gamma \int_0^t v_k(s, Z_s)^{1+\beta} \, ds \]

Itô’s formula yields

\[
\begin{align*}
\frac{dC_t^k}{dt} &= |\dot{x}(t)|^p \, dt + x(t)^p \, A(dt) + px(t)^{p-1} \dot{x}(t)V_t \, dt + x(t)^p \, dV_t \\
&= \left( \frac{d}{dt} |\dot{x}(t)|^p + px(t)^{p-1} \dot{x}(t)V_t + \gamma x(t)^p V_t^{1+\beta} \right) \, dt + x(t)^p \, dM_t^k \\
&= \phi \left( |\dot{x}(t)|, x(t)v_k(t, Z_t)^{1/(p-1)} \right) \, dt + x(t)^p \, dM_t^k,
\end{align*}
\]

where, in the last step, we have used the relations \(1 + \beta = p/(p-1)\) and \(\gamma = \beta^{-1} = p - 1\)
Using (3), we obtain in the limit $t \uparrow T$ that

$$
\int_0^T |\dot{x}(s)|^p \, ds + \int_{[0,T]} x(s)^p \, A(ds) - x_0^p v_k(0, z)
$$

$$
= C_T^k - C_0^k
$$

$$
= \int_0^T \phi(|\dot{x}(t)|, x(t)v_k(t, Z_t)^{1/(p-1)}) \, dt + \int_0^T x(t)^p \, dM_t^k.
$$

Taking expectations yields

$$
E_{0,z} \left[ \int_0^T |\dot{x}(s)|^p \, ds + \int_{[0,T]} x(s)^p \, A(ds) \right]
$$

$$
= x_0^p v_k(0, z) + E_{0,z} \left[ \int_0^T \phi(|\dot{x}(t)|, x(t)v_k(t, Z_t)^{1/(p-1)}) \, dt \right].
$$

Then argue that we may pass to the limit $k \uparrow \infty$ on the right-hand side.
Second step: The more difficult part of the proof is to show that

\[ x^*(t) := x_0 \exp \left( - \int_0^t v_\infty(s, Z_s)^\beta \, ds \right) \]

is an admissible strategy, i.e., \( x^*(t) \to 0 \) as \( t \uparrow T \) and that \( x^* \) has a finite cost.

Based on auxiliary results that are of independent interest. E.g.,
Theorem 2 (Estimates for Laplace functionals of J-functionals).
For the superprocess with homogeneous branching rate $\gamma$, let $I_A$ be a $J$-functional. Define moreover the Borel measure $\alpha_{r,z}(ds)$ on $[r,T]$ by

$$
\int f(s) \alpha_{r,z}(ds) = E_{r,z} \left[ \int_{[r,T]} f(s) A(ds) \right]
$$

for bounded measurable $f : [r,T] \to \mathbb{R}$. Then

(6) \hspace{1cm} \mathbb{E}_{r,\delta_z} [ I_A \mid \langle 1, X_t \rangle, r \leq t \leq T ] = \int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt)

and

(7) \hspace{1cm} - \log \mathbb{E}_{r,\delta_z} [ e^{-I_A} ] \leq - \log \mathbb{E}_{r,\delta_z} [ e^{-\int_{[r,T]} \langle 1, X_t \rangle \alpha_{r,z}(dt) } ] =: u(r)

and $u$ solves the ordinary integral equation

$$
u(t) = \alpha_{r,z}([t,T]) - \gamma \int_{t}^{T} u(s)^{1+\beta} ds, \quad r \leq t \leq T.
$$
The case of nonconstant $\eta(\cdot)$

Need a superprocess with Laplace functionals

$$- \log \mathbb{E}_{r,\delta_z} [ e^{-IA} ] = v(r, \cdot)$$

where $v$ solves

$$v(r, z) = E_{r, z} \left[ A[r, T] - \int_r^t v(s, Z_s)^{1+\beta} \frac{1}{\beta \eta(Z_s)^\beta} ds \right]$$

Such superprocesses were constructed in A.S. (1999) by means of the following $h$-transform technique that was also introduced independently by Engl"ander and Pinsky (1999)
Assume that
\[
\frac{1}{c_T} \eta(z) \leq E_{r,z} [\eta(Z_t)] \leq c_T \eta(z), \quad 0 \leq r \leq t \leq T
\]
and define the space-time harmonic function
\[
h(r, z) := E_{r,z} [\eta(Z_T)]
\]
Let \( Z^h = (Z_t, P^h_{r,z}) \) be the \( h \)-transform of \( Z \), i.e.,
\[
P^h_{r,z} [A] = \frac{1}{h(r, z)} E_{r,z} [\eta(Z_T) 1_A], \quad A \in \mathcal{F}_T,
\]
For an additive functional \( B \) of \( Z \) define an additive functional \( B_h \) of \( Z^h \) by
\[
B_h(ds) = \frac{1}{h(s, Z_s)} B(ds).
\]
With this notation,
\[
K_h(ds) = \frac{1}{h(s, Z_s)} K(ds) = \frac{\eta(Z_s)}{\beta h(s, Z_s)} ds
\]
is a bounded and continuous additive functional of \( Z^h \).
For
\[ \psi(z, \xi) = \left( \frac{\xi}{\eta(z)} \right)^{1+\beta} \]
the function
\[ \psi_h(t, z, \xi) = \psi(z, h(t, z)\xi) \]
is bounded in \( t \) and \( z \) for each \( \xi \). Therefore we can define the
\((Z^h, \psi_h, K_h)\)-superprocess \( X^h = (X^h_t, \mathbb{P}^h_{r,\mu}) \). The superprocess \( X \) under \( \mathbb{P}_{r,\mu} \) is then defined as the law of
\[ X_t(dz) := \frac{1}{h(t, z)} X^h_t(dz) \]
Since for any additive functional \( B \) of \( Z \)
\[ E_{r,z}[B[r,t]] = E_{r,z}\left[ \int_{[r,t]} h(s, Z_s) B_h(ds) \right] = E_{r,z}[B_h[r,t] \cdot \eta(Z_T)] \]
\[ = h(r, z) \cdot E^h_{r,z}[B_h[r,t]] \]
one checks that \( X \) has the desired Laplace functionals.
Then get estimates for the Laplace functionals for $X^h$ by comparison to the case of constant branching by using:

**Proposition 1 (Parabolic maximum principle for Laplace eqn).**

Suppose that $A, \tilde{A}$ and $K, \tilde{K}$ are nonnegative additive functionals of $Z$ with $K, \tilde{K}$ continuous. Suppose furthermore that $A[r, t] \leq \tilde{A}[r, t]$ and $K[r, t] \geq \tilde{K}[r, t]$ for $r \leq t \leq T$. Let $v$ and $\tilde{v}$ be finite and nonnegative solutions of the integral equations

$$v(r, z) = E_{r,z} \left[ A[r, T] - \int_r^T v(t, Z_t)^{1+\beta} K(dt) \right]$$

$$\tilde{v}(r, z) = E_{r,z} \left[ \tilde{A}[r, T] - \int_r^T \tilde{v}(t, Z_t)^{1+\beta} \tilde{K}(dt) \right]$$

Then $v \leq \tilde{v}$.

This then gives estimates for the Laplace functionals of $X$ via the “$h$-transform” for superprocesses
Thank you