Mathematical Modeling of Interest Rates: Challenges and New Directions

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Before August 2007

- Before the credit crunch of 2007, interest rates in the market showed typical textbook behavior. For instance:

**Example 1**
A floating rate bond where LIBOR is set in advance and paid in arrears is worth par (=100) at inception.

\[ 100 \cdot \tau_i \cdot L(T_{i-1}, T_i) \]

where \( \tau_i \) is the “length” of the interval \((T_{i-1}, T_i]\), and \( L(T_{i-1}, T_i) \) denotes the LIBOR at \( T_{i-1} \) for maturity \( T_i \).

**LIBOR**: London Inter-Bank Offer Rate, the interest rate that banks charge each other for loans over predefined time horizons. It is officially fixed once a day by the BBA.
Example 2
The forward rate implied by two deposits coincides with the corresponding FRA rate: \( F_{Depo} = F_{FRA} \).

The forward rate implied by the two deposits with maturity \( T_1 \) and \( T_2 \) is defined by:

\[
F_{Depo} = \frac{1}{T_2 - T_1} \left[ \frac{P(0, T_1)}{P(0, T_2)} - 1 \right]
\]

The corresponding FRA rate is the (unique) value of \( K = F_{FRA} \) for which the following swap(let) has zero value at time \( t = 0 \).

\[
L(T_1, T_2) - K
\]
Before August 2007

Example 3
Compounding two consecutive 3m forward LIBOR rates yields the corresponding 6m forward LIBOR rate:

\[
(1 + \frac{1}{4} F_{3m}^1)(1 + \frac{1}{4} F_{3m}^2) = 1 + \frac{1}{2} F_{6m}^3
\]

where

\[
F_{3m}^1 = F(0; 3m, 6m)
\]
\[
F_{3m}^2 = F(0; 6m, 9m)
\]
\[
F_{6m}^3 = F(0; 3m, 9m)
\]
Before and after August 2007

- These textbook relations are examples of the no-arbitrage rules that held in the market until 3 years ago.
- In fact, differences between theoretically-equivalent rates were present in the market, but generally regarded as negligible.
- For instance, deposit rates and OIS rates for the same maturity used to track each other, but keeping a distance (the basis) of a few basis points.
- Then August 2007 arrived, and our convictions began to waver: The liquidity crisis widened the basis between previously-near rates.
- Consider the following graphs ...
Before and after August 2007

- Since the credit crunch of 2007, the LIBOR-OIS basis has been neither deterministic nor negligible:

USD 3m OIS rates vs 3m LIBOR rates
Before and after August 2007

USD 6m OIS rates vs 6m LIBOR rates
Before and after August 2007

EUR 3m EONIA rates vs 3m LIBOR rates
Before and after August 2007

EUR 6m EONIA rates vs 6m LIBOR rates
Before and after August 2007

- Likewise, since August 2007 the basis between different tenor LIBORs has been neither deterministic nor negligible:

**USD 5y swaps: 1m vs 3m**
Before and after August 2007

EUR 5y swaps: 3m vs 6m
Before and after August 2007
What about arbitrage opportunities?

• Let us consider USD market quotes as of March 25 2009:
  • The 3x6 forward rate implied by the 3m and the 6m deposit rates was $F_D = 2.31\%$.
  • The quoted FRA rate was $F_X = 1.26\%$.

• Again, an apparent inconsistency.
• But ...
• What happens if one tries to “arbitrage” the two markets?
• In the good old times, we would exploit the misalignment between deposit and FRA rates by implementing an arbitrage strategy, which has the following features:
  • The initial value of the strategy is 0.
  • The strategy enables to lock in a positive gain in 6m.
• Let us set $T_1 = 3m$ and $T_2 = 6m$. 
Before and after August 2007

What about arbitrage opportunities?

Let us denote by $D(t, T)$ the time-$t$ deposit price for maturity $T$, and by $L(T_1, T_2)$ the LIBOR rate

$$L(T_1, T_2) = \frac{1}{\tau_{1,2}} \left[ \frac{1}{D(T_1, T_2)} - 1 \right]$$

The arbitrage strategy is as follows. At time $t = 0$:

a) Buy $(1 + \tau_{1,2}F_D)$ deposits with maturity $T_2$, paying

$$(1 + \tau_{1,2}F_D)D(0, T_2) = D(0, T_1) \text{ dollars}$$

b) Sell 1 deposit with maturity $T_1$, receiving $D(0, T_1)$ dollars;

c) Enter a (payer) FRA, paying out at time $T_1$

$$\frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2}L(T_1, T_2)}$$

The value of this strategy at time 0 is zero.
Before and after August 2007
What about arbitrage opportunities?

- At time $T_1$, b) plus c) yields

$$\frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2}L(T_1, T_2)} - 1 = -\frac{1 + \tau_{1,2}F_X}{1 + \tau_{1,2}L(T_1, T_2)} < 0$$

- To pay this, we then sell the $T_2$-deposits, remaining with

$$\frac{1 + \tau_{1,2}F_D}{1 + \tau_{1,2}L(T_1, T_2)} - \frac{1 + \tau_{1,2}F_X}{1 + \tau_{1,2}L(T_1, T_2)} = \frac{\tau_{1,2}(F_D - F_X)}{1 + \tau_{1,2}L(T_1, T_2)} > 0$$

at $T_1$, which is equivalent to $\tau_{1,2}(F_D - F_X)$ received at $T_2$.

- This is clearly an arbitrage. (Too good to be true!)

- In fact, there are issues we can no longer neglect:
  - i) Possibility of default before $T_2$ of the counterparty we lent money to;
  - ii) Possibility of liquidity crunch at times 0 or $T_1$;
  - iii) Regulatory requirements, etc.
Explaining the divergence of “equivalent” rates
Introducing a simple credit model

- Let us denote by $\tau_t$ the default time of the generic interbank counterparty at time $t$.
- We assume independence between default and rates and zero recovery.
- The value at time $t$ of a deposit starting at $t$ and with maturity $T$ is

$$D(t, T) = E\left[e^{-\int_t^T r(u) du} 1_{\{\tau_t > T\}} | \mathcal{F}_t\right]$$

$$= P(t, T) E\left[1_{\{\tau_t > T\}} | \mathcal{F}_t\right]$$

where
- $r$ is the default-free instantaneous rate
- $P(t, T)$ is the price of a default-free zero-coupon bond
Explaining the divergence of “equivalent” rates

• Setting

\[ Q(t, T) := E \left[ 1_{\{\tau_t > T\}} \big| \mathcal{F}_t \right], \]

the LIBOR rate \( L(T_1, T_2) \) is given by

\[
L(T_1, T_2) = \frac{1}{\tau_{1,2}} \left[ \frac{1}{D(T_1, T_2)} - 1 \right]
= \frac{1}{\tau_{1,2}} \left[ \frac{1}{P(T_1, T_2)} \frac{1}{Q(T_1, T_2)} - 1 \right].
\]

• Consider a \( T_1 \times T_2 \) FRA and denote by \( F_X \) its FRA rate.
• Assuming no counterparty risk, its time-0 value can be written as

\[ 0 = E \left[ e^{- \int_0^{T_1} r(u) \, du} \frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2}L(T_1, T_2)} \right] \]
Explaining the divergence of “equivalent” rates

• Carrying out the calculations, we have:

\[
0 = E\left[ e^{-\int_0^{T_1} r(u)\,du} \frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2}L(T_1, T_2)} \right]
\]

\[
= E\left[ e^{-\int_0^{T_1} r(u)\,du} \left(1 - \frac{1 + \tau_{1,2}F_X}{1 + \tau_{1,2}L(T_1, T_2)}\right) \right]
\]

\[
= E\left[ e^{-\int_0^{T_1} r(u)\,du} \left(1 - (1 + \tau_{1,2}F_X)P(T_1, T_2)Q(T_1, T_2)\right) \right]
\]

\[
= P(0, T_1) - (1 + \tau_{1,2}F_X)P(0, T_2)E\left[ Q(T_1, T_2) \right]
\]

• Therefore, the value of the FRA rate \( F_X \) is:

\[
F_X = F_{FRA}(0; T_1, T_2) = \frac{1}{\tau_{1,2}^2} \left[ \frac{P(0, T_1)}{P(0, T_2)} \frac{1}{E\left[ Q(T_1, T_2) \right]} - 1 \right]
\]
Explaining the divergence of “equivalent” rates

- Let us now assume that the OIS swap curve is a good proxy for the risk-free curve.
- The OIS forward rates are then given by

\[ F_{OIS}(0; T_1, T_2) = \frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} - 1 \right] \]

- We can easily see that the FRA rate can be (arbitrarily) higher than the corresponding forward OIS rate.
- In fact,

\[ \frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} \cdot \frac{1}{E[Q(T_1, T_2)]} - 1 \right] > \frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} - 1 \right] \]

since \( 0 < Q(T_1, T_2) < 1 \).
Explaining the divergence of “equivalent” rates

• Similarly, the forward rate implied by the two deposits $D(0, T_1)$ and $D(0, T_2)$, i.e.

$$F_{Depo}(0; T_1, T_2) = \frac{1}{\tau_{1,2}} \left[ \frac{D(0, T_1)}{D(0, T_2)} - 1 \right]$$

$$= \frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} \frac{Q(0, T_1)}{Q(0, T_2)} - 1 \right]$$

will be larger than the FRA rate $F_X$ if

$$\frac{Q(0, T_1)}{Q(0, T_2)} > \frac{1}{E[Q(T_1, T_2)]}$$

• This happens when the market expectation for the credit premium in $L(T_1, T_2)$, inversely $\propto Q(T_1, T_2)$, is low compared to the value implied by $Q(0, T_1)$ and $Q(0, T_2)$. 
A recap of formulas under our simple credit model

<table>
<thead>
<tr>
<th>Rate</th>
<th>Classic formulas</th>
<th>Model formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{FRA}(0; T_1, T_2)$</td>
<td>$\frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} - 1 \right]$</td>
<td>$\frac{1}{\tau_{1,2}} \left[ \frac{P(0, T_1)}{P(0, T_2)} \frac{1}{E[Q(T_1, T_2)]} - 1 \right]$</td>
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<td>$F_{OIS}(0; T_1, T_2)$</td>
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</tr>
</tbody>
</table>
General considerations

A practitioner’s approach

• The previous analysis provides a simple theoretical justification for the divergence of rates that used to be equivalent.

• Such rates, in fact, become “compatible" as soon as credit (and/or liquidity and/or regulatory) risks are taken into account.

• However, instead of resorting to hybrid modeling, practitioners deal with the discrepancies above by segmenting market rates.

• This results in the construction of different zero-coupon curves for different market rate tenors: 1m, 3m, 6m, 1y, ...

• Besides these forward (or projection or growth) LIBOR curves, one also constructs one, or more, discount curves.
General considerations
A global look at market rates

- We no longer have a single zero-coupon curve for each given currency.

EUR market rates (as of 17 October 2011)
General considerations

Construction of forward curves

• Banks construct different zero-coupon curves for each market rate tenor: 1m, 3m, 6m, 1y, ...

Example: The EUR 6m curve
General considerations

The practice of OIS discounting

- Under CSA, it is market practice to use OIS discounting:
  - Since June 2010, USD, Euro and GBP trades in SwapClear (LCH.Clearnet) have been revalued using OIS discounting.
  - Since September 2010, swaption prices in the London inter-dealer option market have been quoted on a forward basis for a number of European currencies.
Definitions in the multi-curve world

Discount curve

- We assume OIS discounting.
- Given a tenor $x$ and an associated time structure $\mathcal{T}^x = \{0 < T^x_0, \ldots, T^x_{M_x}\}$, with $T^x_i - T^x_{i-1} = x$, $i = 1, \ldots, M_x$, OIS forward rates are defined as in the classic single-curve paradigm:

$$F^x_i(t) := F_D(t; T^x_{i-1}, T^x_i) = \frac{1}{\tau^x_i} \left[ \frac{P_D(t, T^x_{i-1})}{P_D(t, T^x_i)} - 1 \right],$$

for $i = 1, \ldots, M_x$, where
  - $\tau^x_i$ is the year fraction for the interval $(T^x_{i-1}, T^x_i]$;
  - $P_D(t, T)$ denotes the discount factor at time $t$ for maturity $T$ for the discount (OIS) curve.
- Consistently with OIS discounting, we assume that our pricing measures are defined by the OIS discount curve.
Definitions in the multi-curve world

Forward LIBOR rate

- The forward LIBOR rate at time $t$ for the period $[T_{i-1}^x, T_i^x]$ is denoted by $L_i^x(t)$ and defined by

$$L_i^x(t) = E_D^{T_i^x} [L^x(T_{i-1}^x, T_i^x)|\mathcal{F}_t],$$

where

- $L^x(T_{i-1}^x, T_i^x)$ denotes the LIBOR set at $T_{i-1}^x$ with maturity $T_i^x$;
- $E_D^{T_i}$ denotes expectation under the $T$-forward measure associated with the discount (OIS) curve;
- $\mathcal{F}_t$ denotes the “information” available at time $t$.

- As in single-curve modeling, $L_i^x(t)$ is the fixed rate to be exchanged at time $T_i^x$ for $L^x(T_{i-1}^x, T_i^x)$ so that the swaplet has zero value at time $t$:

\[
\begin{array}{ccc}
\text{Value swaplet:} & 0 & L^x(T_{i-1}^x, T_i^x) - L_i^x(t) \\
\text{Time:} & t & T_{i-1}^x & T_i^x
\end{array}
\]
Definitions in the multi-curve world

Forward LIBOR rate

The previous definition of forward rate in a multi-curve set up is natural for the following reasons:

1. \( L^x_i(t) = E^{T^x_i}_D [L^x(T^x_{i-1}, T^x_i)|\mathcal{F}_t] \) coincides with the classically defined forward rate in the limit case of a single curve:
   \[
   E^{T^x_i}_D [L^x(T^x_{i-1}, T^x_i)|\mathcal{F}_t] = E^{T^x_i}_D [F_D(T^x_{i-1}; T^x_{i-1}, T^x_i)|\mathcal{F}_t] = F_D(t; T^x_{i-1}, T^x_i)
   \]

2. The rate \( L^x_i(T^x_{i-1}) \) coincides with the LIBOR rate \( L^x(T^x_{i-1}, T^x_i) \):
   \[
   L^x_i(T^x_{i-1}) = E^{T^x_i}_D [L^x(T^x_{i-1}, T^x_i)|\mathcal{F}_{T^x_{i-1}}] = L^x(T^x_{i-1}, T^x_i)
   \]

3. The time-0 value \( L^x_i(0) \) can be stripped from market data.

4. The rate \( L^x_i(t) \) is a martingale under the corresponding OIS forward measure \( Q^{T^x_i} \).

5. This definition allows for a natural extension of the market formulas for swaps, caps and swaptions.
The valuation of interest rate swaps (IRSs) 
(under the assumption of distinct forward and discount curves)

- Given times $T^x_a, \ldots, T^x_b$, consider an IRS whose floating leg pays at each $T^x_k$ the LIBOR rate with tenor $T^x_k - T^x_{k-1} = x$, which is set (in advance) at $T^x_{k-1}$, i.e.

  $$\tau^x_k L^x(T^x_{k-1}, T^x_k)$$

  where $\tau^x_k$ denotes the year fraction.

- The time-$t$ value of this payoff is:

  $$FL(t; T^x_{k-1}, T^x_k) = \tau^x_k P_D(t, T^x_k) E^T_{D_k} \left[ L^x(T^x_{k-1}, T^x_k) | \mathcal{F}_t \right] =: \tau^x_k P_D(t, T^x_k) L^x_k(t)$$

- The swap’s fixed leg is assumed to pay the fixed rate $K$ on dates $T^S_c, \ldots, T^S_d$, with year fractions $\tau^S_j$. 
The valuation of interest rate swaps

- The IRS value to the fixed-rate payer is given by

$$\text{IRS}(t, K) = \sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)$$

- We can then calculate the corresponding forward swap rate as the fixed rate $K$ that makes the IRS value equal to zero at time $t$:

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

- In particular, at $t = 0$ we get:

$$S_{0,b,0,d}^x(0) = \frac{\sum_{k=1}^{b} \tau_k^x P_D(0, T_k^x) L_k^x(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}$$

where $L_1^x(0)$ is the first floating payment (known at time 0).
The valuation of interest rate swaps

- In practice, this swap rate formula can be used to bootstrap the rates $L_k^X(0)$.
- The bootstrapped $L_k^X(0)$ can then be used to price other swaps based on the given tenor.

<table>
<thead>
<tr>
<th>Swap rate</th>
<th>Formulas</th>
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</table>
| OLD       | \[
\sum_{k=1}^{b} \tau_k^X P(0,T_k^X) F_k^X(0) \quad \frac{1 - P(0,T_d^S)}{\sum_{j=1}^{d} \tau_j^S P(0,T_j^S)}
\] |
| NEW       | \[
\sum_{k=1}^{b} \tau_k^X P_D(0,T_k^X) L_k^X(0) \quad \frac{1}{\sum_{j=1}^{d} \tau_j^S P_D(0,T_j^S)}
\] |
The valuation of caplets

• Let us consider a caplet paying out at time $T_{k}^x$

$$
\tau_k^x [L^x(T_{k-1}^x, T_k^x) - K] \geq 0.
$$

• The caplet price at time $t$ is given by:

$$
\text{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) E_D^{T_k^x} \left\{ [L^x(T_{k-1}^x, T_k^x) - K] \geq 0 | \mathcal{F}_t \right\}
$$

$$
= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x} \left\{ [L_k^x(T_{k-1}^x) - K] \geq 0 | \mathcal{F}_t \right\}
$$

• The rate $L_k^x(t) = E_D^{T_k} [L^x(T_{k-1}^x, T_k^x) | \mathcal{F}_t]$ is, by definition, a martingale under $Q_D^{T_k}$.

• Assume that $L_k^x$ follows a (driftless) geometric Brownian motion under $Q_D^{T_k}$.

• Straightforward calculations lead to a (modified) Black formula for caplets.
The valuation of European swaptions

- A payer swaption gives the right to enter at time $T_a^x = T_c^S$ an IRS with payment times for the floating and fixed legs given by $T_{a+1}^x, \ldots, T_b^x$ and $T_{c+1}^S, \ldots, T_d^S$, respectively.
- Therefore, the swaption payoff at time $T_a^x = T_c^S$ is

$$\left[ S_{a,b,c,d}^x(T_a^x) - K \right]^+ \sum_{j=c+1}^d \tau_j^S P_D(T_c^S, T_j^S),$$

where $K$ is the fixed rate and

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^b \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^d \tau_j^S P_D(t, T_j^S)}$$

- We set

$$C_{c,d}^D(t) = \sum_{j=c+1}^d \tau_j^S P_D(t, T_j^S)$$
The valuation of swaptions

- The swaption payoff is conveniently priced under the swap measure $Q_{D}^{c,d}$, whose associated numeraire is $C_{D}^{c,d}(t)$:

$$PS(t, K; T_{a}^{x}, \ldots, T_{b}^{x}, T_{c+1}^{S}, \ldots, T_{d}^{S}) = \sum_{j=c+1}^{d} \tau_{j}^{S}P_{D}(t, T_{j}^{S})$$

$$E^{Q_{D}^{c,d}} \left\{ \left[ S_{a,b,c,d}^{x}(T_{a}^{x}) - K \right]^{+} \sum_{j=c+1}^{d} \tau_{j}^{S}P_{D}(T_{c}^{S}, T_{j}^{S}) \right\} | F_{t}$$

$$= \sum_{j=c+1}^{d} \tau_{j}^{S}P_{D}(t, T_{j}^{S}) E^{Q_{D}^{c,d}} \left\{ \left[ S_{a,b,c,d}^{x}(T_{a}^{x}) - K \right]^{+} | F_{t} \right\}$$

- Hence, also in a multi-curve set up, pricing a swaption is equivalent to pricing an option on the underlying swap rate.

- Assuming that $S_{a,b,c,d}^{x}$ is a (lognormal) martingale under $Q_{D}^{c,d}$, we obtain a (modified) Black formula for swaptions.
The new market formulas for caps and swaptions

<table>
<thead>
<tr>
<th>Type</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>OLD Cplt</td>
<td>( \tau_k^x P(t, T_k^x) \text{Bl}(K, F_k^x(t), \nu_k \sqrt{T_{k-1}^x - t}) )</td>
</tr>
<tr>
<td>NEW Cplt</td>
<td>( \tau_k^x P_D(t, T_k^x) \text{Bl}(K, L_k^x(t), \bar{\nu}<em>k \sqrt{T</em>{k-1}^x - t}) )</td>
</tr>
<tr>
<td>OLD PS</td>
<td>( \sum_{j=c+1}^{d} \tau_j^S P(t, T_j^S) \text{Bl}(K, S_{OLD}(t), \nu \sqrt{T_a^x - t}) )</td>
</tr>
<tr>
<td>NEW PS</td>
<td>( \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) \text{Bl}(K, S_{a,b,c,d}^x(t), \bar{\nu} \sqrt{T_a^x - t}) )</td>
</tr>
</tbody>
</table>
Pricing general interest rate derivatives

- We just showed how to value swaps, caps, and swaptions under the assumption of distinct discount (OIS) and forward curves.

- What about exotics?

- The pricing of general interest rate derivatives should be consistent with the practice of using OIS discounting. In fact:
  - A Bermudan swaption should be more expensive than the underlying European swaptions. In addition, on the last exercise date, a Bermudan swaption becomes a European swaption.
  - A one-period ratchet is equal to a caplet.
  - Etc ...

- We must forsake the traditional single-curve models and switch to a multi-curve framework.
How do we build a multi-curve model?

- Interest-rate multi-curve modeling is based on modeling the joint evolution of a discount (OIS) curve and multiple forward (LIBOR) curves.
- Most banks are currently using a deterministic basis set-up. They choose a model for the OIS curve (short-rate, HJM, LMM, ...), and then build the forward LIBOR curves at a deterministic spread over the OIS curve.
- In general, forward curves can be modeled either directly or indirectly by modeling (possibly stochastic) basis spreads.
- Modeling the OIS curve is necessary for two reasons:
  - Swap rates depend on OIS discount factors.
  - The pricing measures we consider are those defined by the OIS curve.
- Calculating swaption prices in closed form may be hard in general.
The multi-curve LIBOR Market Model (McLMM)

- In the classic (single-curve) LMM, one models the joint evolution of a set of consecutive forward LIBOR rates.
- What about the multi-curve case?
- When pricing a payoff depending on same-tenor LIBOR rates, it is convenient to model rates $L^x_k$.
- This choice is also convenient in the case of a swap-rate dependent payoff. In fact, we can write:

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^{b} \tau^x_k P_D(t, T^x_k) L^x_k(t)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)} = \sum_{k=a+1}^{b} \omega^x_k(t) L^x_k(t)$$

$$\omega^x_k(t) := \frac{\tau^x_k P_D(t, T^x_k)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)}$$

- But, is the modeling of forward rates $L^x_k$ enough?
The McLMM: Alternative formulations

- In fact, we also need to model the OIS forward rates $F^x_k$, $k = 1, \ldots, M_x$:

$$F^x_k(t) = \frac{1}{\tau^x_k} \left[ \frac{P_D(t, T^x_{k-1})}{P_D(t, T^x_k)} - 1 \right]$$

- Denote by $S^x_k(t)$ the additive basis spread

$$S^x_k(t) := L^x_k(t) - F^x_k(t)$$

- By definition, both $L^x_k$ and $F^x_k$ are martingales under the forward measure $Q^T_{D_k}$, and thus their difference $S^x_k$ is as well.

- The LMM can be extended to the multi-curve case in three different ways by:

  1. Modeling the joint evolution of rates $L^x_k$ and $F^x_k$.
  2. Modeling the joint evolution of rates $L^x_k$ and spreads $S^x_k$.
  3. Modeling the joint evolution of rates $F^x_k$ and spreads $S^x_k$. 
An example of McLMM
Modeling rates $F^K_x$ and $L^K_x$

- Given the set of times $\mathcal{T}^x = \{0 < T^x_0, \ldots, T^x_M\}$, which are compatible with the given tenor $x$, we assume that each rate $L^K_x(t)$ evolves under $Q^{T^x_k}$ according to
  \[dL^K_x(t) = \sigma_k(t)L^K_x(t)\,dZ_k(t), \quad t \leq T^x_{k-1}\]

- Likewise, we assume that
  \[dF^K_x(t) = \sigma^D_k(t)F^K_x(t)\,dZ^D_k(t), \quad t \leq T^x_{k-1}\]

- The drift of $X \in \{L^K_x, F^K_x\}$ under $Q^{T^x_j}_D$ is equal to
  \[
  \text{Drift}(X; Q^{T^x_j}_D) = -\frac{d\langle X, \ln(P_D(\cdot, T^x_k)/P_D(\cdot, T^x_j)) \rangle_t}{dt}
  \]
  where $\langle \cdot, \cdot \rangle_t$ denotes instantaneous covariation at time $t$. 
An example of McLMM

- Let us assume $j < k$. The case $j > k$ is analogous.
- The log of the ratio of the two numeraires can be written as

$$\ln \frac{P_D(t, T_k^x)}{P_D(t, T_j^x)} = \ln \frac{1}{\prod_{h=j+1}^{k} (1 + \tau_h^x F_h^x(t))} = - \sum_{h=j+1}^{k} \ln (1 + \tau_h^x F_h^x(t))$$

- We thus get

$$\text{Drift}(L_k^x; Q^T_D) = - \frac{d\langle L_k^x, \ln (P_D(\cdot, T_k^x)/P_D(\cdot, T_j^x))\rangle_t}{dt}$$

$$= \sum_{h=j+1}^{k} \frac{d\langle L_k^x, \ln (1 + \tau_h^x F_h^x)\rangle_t}{dt}$$

$$= \sum_{h=j+1}^{k} \frac{\tau_h^x}{1 + \tau_h^x F_h^x(t)} \frac{d\langle L_k^x, F_h^x\rangle_t}{dt}$$
An example of McLMM

Dynamics under a general forward measure

Proposition. The dynamics of $L^x_k$ and $F^x_k$ under $Q^T_j$ are:

\[
\begin{align*}
\text{j < k :} & \\
\begin{cases}
dL^x_k(t) &= \sigma_k(t)L^x_k(t) \\
dF^x_k(t) &= \sigma^D_k(t)F^x_k(t) 
\end{cases} \\
& \quad \left[ \sum_{h=j+1}^{k} \frac{\rho^{L,F}_{k,h} \tau^x_{h} \sigma^D_{h}(t)F^x_{h}(t)}{1 + \tau^x_{h}F^x_{h}(t)} \right] dt + dZ^j_k(t) \\
& \quad + \sum_{h=j+1}^{k} \frac{\rho^{D,D}_{k,h} \tau^x_{h} \sigma^D_{h}(t)F^x_{h}(t)}{1 + \tau^x_{h}F^x_{h}(t)} dt + dZ^{j,D}_k(t)
\end{align*}
\]

\[
\begin{align*}
\text{j = k :} & \\
\begin{cases}
dL^x_k(t) &= \sigma_k(t)L^x_k(t) \\
dF^x_k(t) &= \sigma^D_k(t)F^x_k(t) 
\end{cases} \\
& \quad dZ^j_k(t) + dZ^{j,D}_k(t)
\end{align*}
\]

\[
\begin{align*}
\text{j > k :} & \\
\begin{cases}
dL^x_k(t) &= \sigma_k(t)L^x_k(t) \\
dF^x_k(t) &= \sigma^D_k(t)F^x_k(t) 
\end{cases} \\
& \quad \left[ - \sum_{h=k+1}^{j} \frac{\rho^{L,F}_{k,h} \tau^x_{h} \sigma^D_{h}(t)F^x_{h}(t)}{1 + \tau^x_{h}F^x_{h}(t)} \right] dt + dZ^j_k(t) \\
& \quad + \sum_{h=k+1}^{j} \frac{\rho^{D,D}_{k,h} \tau^x_{h} \sigma^D_{h}(t)F^x_{h}(t)}{1 + \tau^x_{h}F^x_{h}(t)} dt + dZ^{j,D}_k(t)
\end{align*}
\]
An example of McLMM

Dynamics under the spot LIBOR measure

- The spot LIBOR measure $Q^T_D$ associated with times $\mathcal{T}^x = \{ T_0^x, \ldots, T_{M^x}^x \}$ is the measure whose numeraire is

$$B_D^{T^x}(t) = \frac{P_D(t, T_{\beta(t)-1}^x)}{\prod_{j=0}^{\beta(t)-1} P_D(T_{j-1}^x, T_j^x)},$$

where $\beta(t) = m$ if $T_{m-2}^x < t \leq T_{m-1}^x$, $m \geq 1$, and $T_{-1}^x := 0$.

- Application of the change-of numeraire technique leads to:

$$dL_k^x(t) = \sigma_k(t) L_k^x(t) \left[ \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^L \tau_h^x \sigma_h^D(t) F_h^x(t)}{1 + \tau_h^x F_h^x(t)} \right] dt + dZ_k^d(t)$$

$$dF_k^x(t) = \sigma_k^D(t) F_k^x(t) \left[ \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^D \tau_h^x \sigma_h^D(t) F_h^x(t)}{1 + \tau_h^x F_h^x(t)} \right] dt + dZ_k^{d,D}(t)$$
An example of McLMM

The pricing of caplets

- The pricing of caplets in this multi-curve lognormal LMM is straightforward. We get:

\[
\text{Cplt}(t, K; T_{k-1}^X, T_k^X) = \tau_k^X P_D(t, T_k^X) \text{Bl}(K, L_k^X(t), v_k(t))
\]

where

\[
v_k(t) := \sqrt{\int_t^{T_k^X} \sigma_k(u)^2 \, du}
\]

- As expected, this formula is analogous to that obtained in the single-curve lognormal LMM.

- Here, we just have to replace the “old” forward rates with the corresponding “new” ones, and use the discount factors of the OIS curve.
An example of McLMM

The pricing of swaptions

- Our objective is to derive an analytical approximation for the implied volatility of swaptions.
- To this end, we recall that

\[
S^x_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega^x_k(t) L^x_k(t), \quad \omega^x_k(t) = \frac{\tau^x_k P_D(t, T^x_k)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)}
\]

- Contrary to the single-curve case, the weights \( \omega^x_k(t) \) are not functions of forward LIBOR rates only, since they also depend on discount factors calculated on the OIS curve.
- Therefore we can not write, under \( Q^{c,d}_D \),

\[
dS^x_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \frac{\partial S^x_{a,b,c,d}(t)}{\partial L^x_k(t)} \sigma_k(t) L^x_k(t) dZ^c,d_k(t)
\]
An example of McLMM

The pricing of swaptions

- However, we can resort to a standard approximation technique and freeze the weights $\omega^x_k$ at their time-0 value:

$$S^x_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega^x_k(0)L^x_k(t),$$

thus also freezing the dependence of $S^x_{a,b,c,d}$ on rates $F^x_h$.

- Hence, we can write:

$$dS^x_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega^x_k(0)\sigma_k(t)L^x_k(t)\,dZ^c,d_k(t).$$

- Like in the classic single-curve LMM, we then:
  - Match instantaneous quadratic variations
  - Freeze forward and swap rates at their time-0 value
An example of McLMM

The pricing of swaptions

• This immediately leads to the following (payer) swaption price at time 0:

\[ PS(0, K; T^x_a, \ldots, T^x_b, T^S_{c+1}, \ldots, T^S_d) \]

\[ = \sum_{j=c+1}^{d} \tau^S_j P_D(0, T^S_j) \text{Bl}(K, S_{a,b,c,d}(0), V_{a,b,c,d}), \]

where the swaption volatility (multiplied by \( \sqrt{T^x_a} \)) is given by

\[ V_{a,b,c,d} = \sqrt{\sum_{h,k=a+1}^{b} \frac{\omega^x_h(0)\omega^x_k(0)L^x_h(0)L^x_k(0)\rho_{h,k}}{(S_{a,b,c,d}^x(0))^2} \int_0^{T^x_a} \sigma_h(t)\sigma_k(t) \, dt} \]

• Again, this formula is analogous in structure to that obtained in the single-curve lognormal LMM.
A different class of McLMMs
A framework for the single-tenor case

- In the previous example, we modeled the joint evolution of rates $F^x_k$ and $L^x_k$.
- A different class of McLMMs can be constructed by defining the joint evolution of rates $F^x_k$ and spreads $S^x_k$ under the spot LIBOR measure $Q^{T_x}_D$.
- A single-tenor framework is based on assuming that, under $Q^{T_x}_D$, OIS rates follow general SLV processes:

$$
\begin{align*}
\text{d}F^x_k(t) &= \phi^F_k(t, F^x_k(t))\psi^F_k(V^F(t)) \\
&\quad \cdot \left[ \sum_{h=\beta(t)}^{k} \frac{\tau^x_h \rho_{h,k} \phi^F_h(t, F^x_h(t))\psi^F_h(V^F(t))}{1 + \tau^x_h F^x_h(t)} \text{d}t + \text{d}Z^T_k(t) \right] \\
\text{d}V^F(t) &= a^F(t, V^F(t)) \text{d}t + b^F(t, V^F(t)) \text{d}W^T(t)
\end{align*}
$$
A different class of McLMMs
A general framework for the single-tenor case

where

- $\phi^F_k, \psi^F_k, a^F$ and $b^F$ are deterministic functions of their respective arguments
- $Z^T = \{Z^T_1, \ldots, Z^T_{M_x}\}$ is an $M_x$-dimensional $Q^T_D$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M_x}$
- $W^T$ is a $Q^T_D$-Brownian motion whose instantaneous correlation with $Z^T_k$ is denoted by $\rho^x_k$ for each $k$.
- The stochastic volatility $V^F$ is assumed to be a process common to all OIS forward rates.
- We assume that $V^F(0) = 1$.

Generalizations can be considered where each rate $F^x_k$ has a different volatility process.
A different class of McLMMs
A general framework for the single-tenor case

- We then assume that also the spreads $S_k^x$ follow SLV processes.
- For computational convenience, we assume that spreads and their volatilities are independent of OIS rates.
- This implies that each $S_k^x$ is a $Q^T_D$-martingale as well.
- Finally, the global correlation matrix that includes all cross correlations is assumed to be positive semidefinite.

**Remark.** Several are the examples of dynamics that can be considered. Obvious choices include combinations (and permutations) of geometric Brownian motions and of the stochastic-volatility models of Hagan *et al.* (2002) and Heston (1993). However, the discussion that follows is rather general and requires no dynamics specification.
A different class of McLMMs

Caplet pricing

- Let us consider the $x$-tenor caplet paying out at time $T_k^x$

$$
\tau_k^x [L_k^x(T_k^{x-1}) - K]^+
$$

- Our assumptions on the discount curve imply that the caplet price at time $t$ is given by

$$
\text{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) E_D^{T_k^x} \left\{ [L_k^x(T_k^{x-1}) - K]^+ | \mathcal{F}_t \right\}
= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x} \left\{ [F_k^x(T_k^{x-1}) + S_k^x(T_k^{x-1}) - K]^+ | \mathcal{F}_t \right\}
$$

- Assume we explicitly know the $Q_D^{T_k^x}$-densities $f_{S_k^x(T_k^{x-1})}$ and $f_{F_k^x(T_k^{x-1})}$ (conditional on $\mathcal{F}_t$) of $S_k^x(T_k^{x-1})$ and $F_k^x(T_k^{x-1})$, respectively, and/or the associated caplet prices.
A different class of McLMMs

Caplet pricing

• Thanks to the independence of the random variables $F^x_k(T^x_{k-1})$ and $S^x_k(T^x_{k-1})$ we equivalently have:

$$\text{Cplt}(t, K; T^x_{k-1}, T^x_k) \tau^x_k P_D(t, T^x_k)$$

$$= \int_{-\infty}^{+\infty} E^{T^x_k}_D \{ [F^x_k(T^x_{k-1}) - (K - z)]^+ | \mathcal{F}_t \} f_{S^x_k(T^x_{k-1})}(z) \, dz$$

$$= \int_{-\infty}^{+\infty} E^{T^x_k}_D \{ [S^x_k(T^x_{k-1}) - (K - z)]^+ | \mathcal{F}_t \} f_{F^x_k(T^x_{k-1})}(z) \, dz$$

• One may use the first or the second formula depending on the chosen dynamics for $F^x_k$ and $S^x_k$:
  • Lognormal
  • Heston
  • SABR
  • ...

A different class of McLMMs

Caplet pricing

- To calculate the caplet price one needs to derive the dynamics of $F^X_k$ and $V^F$ under the forward measure $Q^{T^X_k}_D$.
- Notice that the $Q^{T^X_k}_D$-dynamics of $S^X_k$ and its volatility are the same as those under $Q^T_D$ thanks to our independence assumption.
- To this end, we apply the standard change-of-numeraire result, which in this case reads as:

$$\text{Drift}(X; Q^{T^X_k}_D) = \text{Drift}(X; Q^T_D) + \frac{d\langle X, \ln \frac{P_D(\cdot, T^X_k)}{P_D(\cdot, T^X_k \beta(t)-1)} \rangle_t}{dt}$$

where $X$ is a continuous process, and $\langle \cdot, \cdot \rangle_t$ denotes again instantaneous covariation at time $t$. 
A different class of McLMMs

Caplet pricing

- The dynamics of $F^x_k$ and $V^F$ under $Q_{D}^{T^x_k}$ are thus given by:

$$
\begin{align*}
\mathrm{d}F^x_k(t) &= \phi^F_k(t, F^x_k(t))\psi^F_k(V^F(t)) \, \mathrm{d}Z^k(t) \\
\mathrm{d}V^F(t) &= a^F(t, V^F(t)) \, \mathrm{d}t + b^F(t, V^F(t)) \\
&\quad \cdot \left[ - \sum_{h=\beta(t)}^{k} \frac{\tau^x_h \phi^F_h(t, F^x_h(t))\psi^F_h(V^F(t))\rho^x_h}{1 + \tau^x_h F^x_h(t)} \, \mathrm{d}t + \mathrm{d}W^k(t) \right]
\end{align*}
$$

where $Z^k$ and $W^k$ are $Q_{D}^{T^x_k}$-Brownian motions.

- By resorting to standard drift-freezing techniques, one can find tractable approximations of $V^F$ for typical choices of $a^F$ and $b^F$, which will lead either to an explicit density $f_{F^x_k(T^x_{k-1})}$ or to an explicit option pricing formula (on $F^x_k$).

- This, along with the assumed tractability of $S^x_k$, will finally allow the calculation of the caplet price.
A different class of McLMMs

Swaption pricing

• Let us consider a (payer) swaption, which gives the right to enter at time $T^x_a = T^S_c$ an interest-rate swap with payment times for the floating and fixed legs given by $T^x_{a+1}, \ldots, T^x_b$ and $T^S_{c+1}, \ldots, T^S_d$, respectively, with $T^x_b = T^S_d$ and where the fixed rate is $K$.

• The swaption payoff at time $T^x_a = T^S_c$ is given by

$$
\left[ S^x_{a,b,c,d}(T^x_a) - K \right]^+ \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j),
$$

where the forward swap rate $S^x_{a,b,c,d}(t)$ is given by

$$
S^x_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau^x_k P_D(t, T^x_k)L^x_k(t)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)}.
$$
A different class of McLMMs

Swaption pricing

• The swaption payoff is conveniently priced under $Q_D^{c,d}$:

$$PS(t, K; T_a^x, \ldots, T_b^x, T_{c+1}^S, \ldots, T_d^S)$$

$$= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) E_D^{c,d} \left\{ \left[ S_{a,b,c,d}^x(T_a^x) - K \right]^+ \mid \mathcal{F}_t \right\}$$

• To calculate the last expectation, we set

$$\omega_k^x(t) := \frac{\tau_k^x P_D(t, T_k^x)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

and write:

$$S_{a,b,c,d}^x(t) = \sum_{k=a+1}^{b} \omega_k^x(t) L_k^x(t)$$

$$= \sum_{k=a+1}^{b} \omega_k^x(t) F_k^x(t) + \sum_{k=a+1}^{b} \omega_k^x(t) S_k^x(t) =: \tilde{F}(t) + \tilde{S}(t)$$
A different class of McLMMs
Swaption pricing

- The processes $S_{a,b,c,d}^x$, $\bar{F}$ and $\bar{S}$ are all $Q_{D}^{c,d}$-martingales.
- $\bar{F}$ is equal to the classic single-curve forward swap rate that is defined by OIS discount factors, and whose reset and payment times are given by $T_c^S, \ldots, T_d^S$.
- If the dynamics of rates $F_k^x$ are sufficiently tractable, we can approximate $\bar{F}(t)$ by a driftless stochastic-volatility process, $\tilde{F}(t)$, of the same type as that of $F_k^x$.
- The process $\bar{S}$ is more complex, since it explicitly depends both on OIS discount factors and on basis spreads.
- However, we can resort to a standard approximation and freeze the weights $\omega_k^x$ at their time-0 value, thus removing the dependence of $\bar{S}$ on OIS discount factors.
A different class of McLMMs
Swaption pricing

• We then assume we can further approximate $\tilde{S}$ with a dynamics $\tilde{S}$ similar to that of $S_k^x$, for instance by matching instantaneous variations.

• After the approximations just described, the swaption price becomes

$$PS(t, K; T_a^x, \ldots, T_b^x, T_{c+1}^S, \ldots, T_d^S)$$

$$= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) E_D^{c,d} \{ [\tilde{F}(T_a^x) + \tilde{S}(T_a^x) - K]^+ | F_t \}$$

which can be calculated exactly in the same fashion as the previous caplet price.
A general recipe for multi-curve modeling

A simple stochastic-basis framework

• In general, is there any easy-to-use recipe for multi-curve modeling, i.e. for jointly modeling discount and forward curves at a stochastic spread over one another?
• The answer is yes. We can introduce a general stochastic-basis framework that can be plugged in on top of any term structure model for the OIS curve.
• Let us assume that the OIS curve evolution is described by some classic single-curve model.
• We model additive LIBOR-OIS basis spreads

\[ S_i^X(t) = L_i^X(t) - F_i^X(t) \]

• \( L_i^X \) and \( F_i^X \) are both martingales under the OIS \( T_i^X \)-forward measure \( Q_{T_i^X}^{D_i} \). Hence \( S_i^X \) is a \( Q_{T_i^X}^{D_i} \)-martingale as well.
A simple stochastic basis framework

• Our stochastic-basis framework is based on assuming that

\[ S_i^X(t) = \rho_i^X F_i^X(t) + \alpha_i^X [(1 - \lambda_i^X) \chi_i^X(t) + \lambda_i^X] \]

where

• \( \alpha_i^X := S_i^X(0) - \rho_i^X F_i^X(0) \);
• \( \rho_i^X \)'s and \( \lambda_i^X \)'s are constant parameters;
• The basis factors \( \chi_i^X \) are independent of OIS rates;
• \( \chi_i^X(0) = 1 \).

• LIBORs are then explicitly given by

\[
L_i^X(t) = F_i^X(t) + \rho_i^X F_i^X(t) + \alpha_i^X [(1 - \lambda_i^X) \chi_i^X(t) + \lambda_i^X] \\
= (1 + \rho_i^X) F_i^X(t) + \alpha_i^X [(1 - \lambda_i^X) \chi_i^X(t) + \lambda_i^X]
\]

• Swap rates can be calculated by using the new market formula.
A simple stochastic basis framework

Terminal correlations

- Using the independence of $\mathcal{X}_i^x(t)$ and $F_i^x(t)$, it is trivial to calculate the following (terminal) $T_i^x$-forward correlations:

  \[
  \text{Corr}(F_i^x(t), S_i^x(t)) = 1_{\{\rho_i^x \neq 0\}} \frac{\text{sign}(\rho_i^x)}{\sqrt{1 + \frac{(\alpha_i^x)^2(1 - \chi_i^x)^2}{(\rho_i^x)^2} \frac{\text{Var}[\mathcal{X}_i^x(t)]}{\text{Var}[F_i^x(t)]}}} 
  \]

  \[
  \text{Corr}(F_i^x(t), L_i^x(t)) = 1_{\{\rho_i^x \neq -1\}} \frac{\text{sign}(1 + \rho_i^x)}{\sqrt{1 + \frac{(\alpha_i^x)^2(1 - \chi_i^x)^2}{(1 + \rho_i^x)^2} \frac{\text{Var}[\mathcal{X}_i^x(t)]}{\text{Var}[F_i^x(t)]}}} 
  \]

- The terminal correlations between OIS forward rates, LIBORs and spreads corresponding to different time intervals and/or tenors may also be calculated explicitly, depending on the chosen OIS curve model.
Example I: a multi-curve LIBOR Market Model

- Let us consider tenors $x_1 < x_2 < \cdots < x_n$ with associated time structures $\mathcal{T}^{x_i} = \{0 < T_0^{x_i}, \ldots, T_{M_{x_i}}^{x_i}\}$, where $\mathcal{T}^{x_n} \subset \mathcal{T}^{x_{n-1}} \subset \cdots \subset \mathcal{T}^{x_1} =: \mathcal{T}$.

- We assume that, under the OIS spot LIBOR measure $Q^T_D$, the OIS forward rates $F_{k}^{x_1}$, $k = 1, \ldots, M_1$, follow:

$$
\begin{align*}
\text{d}F_{k}^{x_1}(t) &= \sigma_{k}^{x_1} V^F(t) \left[ \frac{1}{T_k^{x_1}} + F_k^{x_1}(t) \right] \left[ V^F(t) \sum_{h=\beta(t)}^{k} \rho_{h,k} \sigma_{h}^{x_1} \text{d}t + \text{d}Z_k^T(t) \right] \\
\text{d}V^F(t) &= \epsilon V^F(t) \text{d}W^T(t)
\end{align*}
$$

where:

- $\sigma_{k}^{x_1}$ is a positive constant, for each $k$;
- $\text{d}Z_k^T(t)\text{d}Z_j^T(t) = \rho_{k,j} \text{d}t$, $k, j = 1, \ldots, M_1$;
- $\text{d}W^T(t)\text{d}Z_k^T(t) = \psi_k \text{d}t$;
- $V^F(0) = 1$;
- $\beta(t) = m$ if $T_{m-2}^{x} < t \leq T_{m-1}^{x}$, $m \geq 1$, and $T_{-1}^{x} := 0$. 
A multi-curve LIBOR Market Model

- The dynamics of forward rates $F^x_k$, for tenors $x \in \{x_2, \ldots, x_n\}$, can be obtained by Ito’s lemma, noting that $F^x_k$ can be written in terms of shorter tenor rates $F^{x_1}_k$ as follows:

$$\prod_{h=i_{k-1}+1}^{i_k} \left[ 1 + \tau^x_{h} F^x_{h}(t) \right] = 1 + \tau^x_k F^x_k(t),$$

for some indices $i_{k-1}$ and $i_k$.

- These dynamics of $F^x_k$ are consistent across different tenors $x$. For instance, both 3m-rates and 6m-rates follow shifted-lognormal processes with common SABR stochastic volatility.

- This allows us to simultaneously price, with the same type of formula, caps and swaptions with different tenors $x$. 
A multi-curve LIBOR Market Model

• The dynamics for the $x$-tenor rate $F^x_k$ under $Q^{T^x_k}_D$ are given by:

\[
\begin{align*}
    dF^x_k(t) &= \sigma^x_k V^F(t) \left[ \frac{1}{\tau^x_k} + F^x_k(t) \right] dZ^k,x(t) \\
    dV^F(t) &= -\epsilon [V^F(t)]^2 \sum_{h=\beta(t)}^{i_k} \sigma^x_1 \psi^x_1 dt + \epsilon V^F(t) dW^{k,x}(t)
\end{align*}
\]

• Regarding basis spreads dynamics in our framework, we set $\rho^x_k = \lambda^x_k = 0$, for all $k$ and $x$, and assume that basis factors $\lambda^x_k$ follow a one-factor geometric Brownian motion:

\[
\begin{align*}
    S^x_k(t) &= S^x_k(0) M(t), \quad k = 1, \ldots, M_x, \\
    dM(t) &= \sigma M(t) dZ(t)
\end{align*}
\]

where $Z$ is a Brownian motion independent of $Z^k,x_k$ and $W^{k,x}$ and $\sigma$ is a positive constant. Clearly, $M^x(0) = 1$. 
A multi-curve LIBOR Market Model
A numerical example

- By the independence of spreads and rates, a caplet price can be calculated as an integral of SABR prices times a normal density (we approximate linearly the drift term of $V^F$).
- The caplet pricing formula we obtain can be used to price caps on any tenor $x$.
- We consider a numerical example based on EUR data as of September 15th, 2010.
- We calibrate the market prices of standard $x = 6m$-tenor caps using the model formula.
- We then price caps on a non-standard tenor $x = 3m$ by using the same formula, and assuming a specific correlation structure $\rho_{i,j}$.
A multi-curve LIBOR Market Model

A numerical example

Figure: Absolute differences (in%) between market and model cap volatilities.
Figure: Absolute differences (in bp) between model-implied 3m-LIBOR cap volatilities and model 6m-LIBOR ones.
Example II: a two-factor multi-curve model

- Let us now assume that the OIS curve follows the one-factor Hull-White (1990) model:

  \[ dr(t) = [\vartheta(t) - ar(t)] \, dt + \sigma(t) \, dW(t) \]

  where \( a \) is a positive constant parameter, and \( \vartheta \) and \( \sigma \) are deterministic functions of time.

- Hence, forward OIS rates are explicitly given by:

  \[ F^x_i(t) = \left[ A(t, T^x_{i-1}, T^x_i) \, e^{-B(t, T^x_{i-1}, T^x_i)r(t)} - 1 \right] / \tau^x_i \]

  where \( A \) and \( B \) are deterministic functions of time.

- Then we assume that basis factors \( X^x_i \) follow a common geometric Brownian motion \( X^x \):

  \[ X^x_i(t) = X^x(t), \quad \text{for each } i \]

  \[ dX^x(t) = \eta^x(t)X^x(t) \, dZ^x(t), \quad X^x(0) = 1, \]

  where \( \eta^x \) is a deterministic function of time, and the Brownian motion \( Z^x \) is independent of OIS rates.
A two-factor multi-curve model
Caplet and swaption pricing

• Caplet and swaption prices can be expressed in terms of a two-dimensional integral of the respective payoff functions times a bivariate normal density.
• Since forward LIBOR and swap rates are affine functions of $\mathcal{X}^x$ conditional on OIS rates, one integration can be carried out explicitly.
• Therefore, caplets and swaptions can be priced semi-analytically by a one-dimensional integration of a closed-form function of Black-Scholes type.
• Let us consider a (payer) swaption paying out at time $T^x_a$:

$$
\left[ S_{a,b,c,d}(T^x_a) - K \right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T^x_a, T^S_j)
$$

• The swaption payoff can be expressed as follows:

$$
\left[ C(y(T^x_a))\mathcal{X}^x(T^x_a) - D(y(T^x_a)) \right]^+
$$
A two-factor multi-curve model
Caplet and swaption pricing

• The payer swaption price at time 0 can be calculated by taking expectation under the $T^x_a$-forward measure:

$$PS(0, K) = P_D(0, T^x_a)E^{T^x_a}\left\{ \left[ C(y(T^x_a)) \chi^x(T^x_a) - D(y(T^x_a)) \right]^+ \right\}$$

$$= P_D(0, T^x_a) \int_{-\infty}^{\infty} h(C(y), D(y), V\chi(T^x_a)) \cdot \left[ \text{normal density of } y(T^x_a) \right] (y) \, dy$$

$$h(A, B, V) := \begin{cases} 
Bl(A, B, V, 1) & \text{if } A, B > 0 \\
Bl(A, B, V, -1) & \text{if } A, B < 0 \\
A - B & \text{if } A \geq 0, B \leq 0 \\
0 & \text{if } A < 0, B \geq 0 
\end{cases}$$

$$Bl(F, K, v, w) := wF\Phi\left( w \frac{\ln \frac{F}{K} + \frac{v^2}{2}}{v} \right) - wK\Phi\left( w \frac{\ln \frac{F}{K} - \frac{v^2}{2}}{v} \right),$$
A two-factor multi-curve model
Extension to a finer time structure

• For pricing purposes one may want to consider a finer time structure than the initial $\mathcal{T}^x$.

• Let us then assume we are given a time structure $\mathcal{T} := \{x < T_1, \ldots, T_M\}$ that contains $\mathcal{T}^x$, with the possible exception of $T_0^x$.

• OIS forward rates at time $t$ for the interval $[T_i - x, T_i]$ can be defined by the classic formula:

\[
F_D(t; T_i - x, T_i) = \frac{1}{x} \left[ \frac{P_D(t, T_i - x)}{P_D(t, T_i)} - 1 \right], \quad i = 1, \ldots, M
\]

• The $x$-tenor forward LIBOR rate at time $t$ for the interval $[T_i - x, T_i]$ can then be defined as follows:

\[
L^x(t; T_i - x, T_i) = (1 + \bar{\rho}_i^x)F_D(t; T_i - x, T_i) + \bar{\alpha}_i^x[(1 - \bar{\lambda}_i^x)\mathcal{X}^x(t) + \bar{\lambda}_i^x]
\]
A two-factor multi-curve model
Bermudan swaptions pricing

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<th>$K$</th>
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<th>3%</th>
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<td>5.84%</td>
<td>4.50%</td>
<td>3.41%</td>
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Table: Prices of 10y-non-call-1y Bermudan (payer) swaptions.
A two-factor multi-curve model

Single looks

Table: Prices of single looks paying out $\max(\text{CMS}_{10y}-\text{CMS}_{2y}-K,0)$ in 5y. Strikes and prices are in bp.
A two-factor multi-curve model

Range accruals

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<th>1%-2%</th>
<th>2%-3%</th>
<th>3%-4%</th>
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<td>52.71%</td>
<td>43.31%</td>
<td>29.36%</td>
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</tbody>
</table>

Table: Prices of 6y-RACLs. Underlying index: 3m LIBOR. Window: 3y-6y.