Interacting Particle Models of Systemic Risk

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Systemic Risk in Banking Networks

- Banking networks pose the most significant source of systemic risk due to their high degree of interconnections and impact on the economy.
- These networks are highly complex, continuously evolve over time and are subject to many ongoing shocks.
- Stochastic modeling of this problem remains open with multiple proposed approaches.
- We propose a new view using methods of Interacting Particle Systems.
- A flexible framework that is convenient for large-scale qualitative analysis while allowing for much granularity.
Modeling Approach

- Focus on the dynamic evolution of the whole network.
- Think of banks as particles or *individuals* that interact with other banks.
- Banks are born, grow over time, and eventually die (i.e. default).
- These mechanisms, especially default, involve mean-field-type interactions that create inter-dependencies.
Related Literature

- Network models: Cont et al. (2011a,b)
- Scaling Limits of Large Portfolio Losses: Giesecke et al. (2011), Cvitanic et al. (2011)
- Large deviations approaches: Giesecke et al. (2011), Papanicolaou (2012)
**Stochastic Model**

- $X_i(t)$ – net assets of bank $i$ at date $t$.
- Each bank is born at epoch $g_i$ and dies at $d_i$.
- Let $l(t) = \{i : g_i \leq t < d_i\}$ be the subset of banks alive at $t$.
- $N(t) = |l(t)|$ is the size of the system.
- $\bar{X}(t) := \frac{1}{N(t)} \sum_{i \in l(t)} X_i(t)$ – average bank size.
- $X(t) \in \bigcup_n \mathbb{R}^n$ is the state process.
System Evolution

- $N(t)$ is a **birth-and-death process** modeled through corresponding event intensities.
- Birth rate $\lambda^+(N(t))$ – larger when $N(t)$ is small (less competition).
- Birth size $\xi_i \sim F$.
- Default rate $\lambda^-(X_i(t))$ – reduced-form credit model; decreases in $X_i$.
- Due to interconnections, defaults affect other banks: $X_i(d_j+) = [1 - \theta(X_j(d_j-), \overline{X}(d_j-))] \cdot X_i(d_j-)$, where $\theta(\cdot)$ is the random proportion of bank $i$ assets reduced due to default of bank $j$.
- Between defaults, assets grow at deterministic rate $r_i$.
- **Piecewise-deterministic** model.
Individual dynamics:

\[ X_i(t) = X_i(g_i) + \int_{g_i}^{t} r_i X_i(s) ds - \sum_{j=1}^{\infty} \theta(\bar{X}(d_j - )) X_i(d_j - ) \mathbf{1}_{\{g_i < d_j \leq t\}} ; \quad g_i \leq t < d_i , \]

Collective dynamics: \( P(t) = \prod_{i \in I(t)} X_i(t) \)

\[ P(t) = P(s) \exp \left[ \rho(s)(t - s) + \sum_{j} (r_j(t - g_j) + \log \xi_j) \mathbf{1}_{\{s < g_j < t\}} \right. \]

\[ \left. + \sum_{\ell} \left\{ (N(d_\ell - ) - 1) \log(1 - \theta(\bar{X}(d_\ell - ))) - r_\ell (t - s) - \log X_\ell(s) \right\} \mathbf{1}_{\{s < d_\ell < t\}} \right] . \]

Illustrative example: \( r_i \equiv r, \lambda^- (x) = \frac{\lambda^-}{1 + x}, \lambda^+ (n) = \frac{\lambda^+}{1 + n}, \)

\( \theta(x_j, \bar{x}) \sim \text{Beta}(\frac{x_j}{\rho \bar{x}}, b) . \)
Sample Path of System Dynamics

**Figure:** Trajectory of mean asset size $\bar{X}(t)$ and normalized number of banks $N(t)/N(0)$. Bottom panel: default times and corresponding impact proportions $\theta(\cdot)$. We take $r = 0.02$, $\lambda^- (x) = 0.05/(1 + x)$, $\lambda^+ (n) = 300/(1 + n)$, $\xi \sim \text{Exp}(1)$ and $N_0 = 100$ and
Model Features

- The system is **self-stabilizing**: when $N(t)$ is large there are more defaults causing $\overline{X}(t)$ to fall; when $N(t)$ is small, birth rate is higher.
- Lifetime of each individual bank is finite.
- High turnover microscopically; stable macroscopically.
- If $\lambda^- (x) > 0$ is bounded away from zero then with positive probability the system will eventually completely collapse and regenerate. Large shocks are intrinsic.
- Numerous **add-ons** are possible: diffusion terms in $X_i(t)$; more correlations; default cascades; inhomogeneous dynamics, etc.
Recurrence

Proposition

*Under technical conditions on $\lambda^-$ and $\theta$, the system has an invariant distribution.*

- Rule out explosion (intrinsic defaults + interaction); births allow regeneration.
- Sufficient condition: $\int_0^{\infty} \lambda^-(x_0 e^{rt}) \, dt = +\infty$ for all $x_0$. 

Systemic Shocks

- While the system will eventually collapse and regenerate, large shocks are very rare.
- We are interested in understanding the mechanism/frequency of such shocks in terms of model ingredients.
- Focus on the case of moderately large $N(0)$ – still far from mean-field limit.
- Large shock $= N(t)$ or $\bar{X}(t)$ "small".
  - e.g. let $\tau = \inf\{t : N(t) < n\}$. Wish to compute:
    - $P(N(T) < n)$;
    - $P(\tau < T)$;
    - Path-distribution of $X(\cdot)$ on $[0, \tau)$.
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Generating Large Shocks

- Think of one realization of $X(\cdot)$ as a "scenario".
- Naive use of Monte Carlo to generate a lot of scenarios to understand behavior of big shocks is very inefficient when the corresponding probabilities are small.
- Variance Reduction is a must.
- Importance Sampling is hard to implement as it’s not clear how to choose a new measure for $X$.
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Empirical Importance Sampling

- Think of each scenario as a "particle".
- Scenarios are propagated forward using a genetic-type algorithm: resampling-mutation steps.
- Particles that get "closer" to events of interest are assigned higher weights and are more likely to spawn children.
- Particles with low weights get culled.
- Work under the original system dynamics.
Consider a collection of $J$ scenario path-particles $X^{(j)}$, $j = 1, \ldots, J$. The particles evolve according to a Feynman-Kac measure change:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{1}{Z_T} \prod_{k=1}^{T} G_k(X)$$

where $G_k$ are the Feynman-Kac potentials on the increasing path spaces of length $k$: $G_k(X) \equiv G_k(X(t_0), X(t_1), \ldots, X(t_k))$.

Sequential algorithm: At dates $t_k$ re-sample particles using weights $w_j := \frac{G_k(X^{(j)})}{\sum_{\ell} G_k(X^{(\ell)})}$.

Propagate particles independently using $\mathbb{P}$-dynamics.
IS Approximation

- Approximate:

\[
\mathbb{E}[f(\mathbf{X}(T))] = \mathbb{E} \left[ f(\mathbf{X}(T)) \prod_{k=1}^{T} G_{k}^{-1}(\mathbf{X}) \prod_{k=1}^{T} G_{k}(\mathbf{X}) \right] \\
= \eta_T(\tilde{f}(\mathbf{X})) \prod_{k=1}^{T} \eta_k(G_k), \quad \tilde{f}(\mathbf{X}) := f(\mathbf{X}(T)) \prod_{k=1}^{T} G_{k}^{-1}(\mathbf{X})
\]

\[
\eta_k(g) = \gamma_k(g)/\gamma_k(1) \quad \gamma_k(g) := \mathbb{E}[g(\mathbf{X}) \prod_{\ell=1}^{k} G_{\ell}(\mathbf{X})].
\]

- Obtain an unbiased estimator based on

\[
\eta_k^{(J)}(G_k) = \frac{1}{J} \sum_{j=1}^{J} G_k(\mathbf{X}^{(j)}).
\]

- Typical potential: 

\[
G_k(\mathbf{X}^{(j)}(t_0), \mathbf{X}^{(j)}(t_1), \ldots \mathbf{X}^{(j)}(t_k)) = \exp(\alpha(\min_{\ell \leq k-1} N(t_\ell) - \min_{\ell \leq k} N(t_\ell))) \quad \text{(multiplicative)}.
\]

- Preference to particles where \(N(t)\) is setting new lows.
Importance Sampling

Figure: Histogram of 400 MC simulations: under original measure (independent scenarios) and using a F-K potential $G_k(X) = \exp(-\alpha(\min_{\ell \leq k} N(\ell) - \min_{\ell < k} N(\ell)))$. 
Algorithm Details

- Sufficient to have just a few hundred particles.
- The crux of the method is in choosing a good potential $G_k$ (in particular the potential strength $\alpha$) – must be adapted to the problem at hand.
- Resample for example every 1 period (in between use exact simulation of the birth-and-death process $N(t)$ and corresponding $X^{(j)}$, $t_k = k$).
- By storing the path genealogies of the particles surviving at $T$ have access to (unbiased) conditional distribution of $X(\cdot)$ before the shock (note: path degeneracy).
- The method dynamically twists the measure while keeping the simulation part very simple.
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Examples of Rare Events

- Probability of a very low number of banks (or very low total bank assets) at a fixed date $T$.
- Probability of a major shock in the next $T$ periods.
- Relationship between low $N(T)$ and low $\bar{X}(T)$.

**Example:**

\[
P(\min_{s\leq 50} N(s) < 80) \approx 0.011 \\
P(\min_{s\leq 50} N(s) < 70) \approx 7.3 \cdot 10^{-5} \\
P(\min_{s\leq 50} N(s) < 60) \approx 1.2 \cdot 10^{-7}. \\
\]
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Observed Effects

- Little correlation between low $N(T)$ versus low $\bar{X}(T)$.
- Low $\bar{X}(T)$ is caused by a single major default ("Lehman").
- Low $N(T)$ is due to ongoing small defaults = "recession".
- Conditional on many defaults: $\bar{X}(T)$ is low, but $N(T)$ remains moderate.
Genealogy of the path-particles

Figure: Three particle genealogies conditional on a high number of defaults.
Work in Progress

- Have a framework for simulating dynamic defaults in an interacting network.
- Understanding the relationships and mechanisms of rare events unlocks many interesting phenomena.
- Another analytic tool is the LLN scaling limit as $N(0) \rightarrow \infty$ (average out the noise from the defaults and births).

THANK YOU!
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THANK YOU!
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