Robust Maximization of Asymptotic Growth under Covariance Uncertainty

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Uncertainty in covariance (or, equivalently, in variance) has been drawing increasing attention.


Most of the literature focuses on the superhedging problem.

We study something different: robust growth-optimal trading.

**The Question**

How to maximize the growth rate of one’s wealth when precise covariance structure of the underlying assets is not known?
Let $E \subset \mathbb{R}^d$ be an open connected set (e.g., “positive octant”), and $S^d$ be the set of $d \times d$ symmetric matrices.

- $X$: price process of $d$ assets taking values in $E$.
- $\theta, \Theta : E \mapsto (0, \infty)$ are functions in $C^{0,\alpha}_{\text{loc}}(E)$ with $\theta < \Theta$ in $E$.
- $C$: set of $C^1$ functions $c : E \mapsto S^d$ s.t.

$$\theta(x)I_d \leq c(x) \leq \Theta(x)I_d, \text{ for all } x \in E. \quad (1)$$

Each $c \in C$ is a covariance structure that might materialize.
If $c \in C$ is a priori given, we define

$(L^{c(\cdot)}f)(x) := \frac{1}{2} \sum_{i,j=1}^{d} c_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{1}{2} \text{Tr}[c(x)D^2 f(x)]$.

$Q^c$: the solution to the (generalized) martingale problem on $E$ for the operator $L^{c(\cdot)}$.

$\Pi^c := \{ \mathbb{P} \mid \mathbb{P} \ll_{\text{loc}} Q^c, \text{X doesn't explode } \mathbb{P}-\text{a.s.} \}$.

[dominated by $Q^c$; drift uncertainty]

In this paper, we consider

$\Pi := \bigcup_{c \in C} \Pi^c$.

[Non-dominated; covariance uncertainty]
We say $\pi \in \mathcal{V}$ is an **admissible trading strategy** if it is a predictable process s.t.

- $\pi$ is $X$-integrable under $Q^c$, for all $c \in C$;
- $V^\pi_t := 1 + \int_0^t \pi'_s dX_s > 0$ $Q^c$-a.s., for all $c \in C$ and $t \geq 0$.

The **asymptotic growth rate** of $V^\pi$ under $P$ is defined as

$$g(\pi; P) := \sup \left\{ \gamma \in \mathbb{R} \left| \liminf_{t \to \infty} \frac{\log V^\pi_t}{t} \geq \gamma \right. \right\}.$$

($\approx \sup \left\{ \gamma \in \mathbb{R} \mid V^\pi_t \geq e^{\gamma t} \text{ as } t \text{ large } P\text{-a.s.} \right\}$)
Our Goal

Choose a $\pi^* \in \mathcal{V}$ s.t. $\mathcal{V}\pi^*$ attains the growth rate

$$\sup_{\pi \in \mathcal{V}} \inf_{P \in \Pi} g(\pi; P)$$

under all $P$ in $\Pi$ (or at least in a large enough subset $\Pi^*$ of $\Pi$).

We call $\sup_{\pi \in \mathcal{V}} \inf_{P \in \Pi} g(\pi; P)$ the robust maximal asymptotic growth rate, which can be considered as the maximal worst-case asymptotic growth rate.
Kardaras & Robertson (2012): for any $D \subset E$ and $\lambda \in \mathbb{R}$, consider

$$H_{\lambda}^c(D) := \{ \eta \in C^2(D) \mid L^c(\cdot)\eta + \lambda \eta = 0, \eta > 0 \text{ in } D \},$$

and define the \textit{principal eigenvalue} for $L^c(\cdot)$ on $D$ as

$$\lambda^{*,c}(D) := \sup\{ \lambda \in \mathbb{R} \mid H_{\lambda}^c(D) \neq \emptyset \}.$$
When \( c \in \mathcal{C} \) is a priori given...

**Theorem [Kardaras & Robertson (2012)]**

Take \( \eta^{*,c} \in H^c_{\lambda^{*,c}(E)}(E) \) and define

\[
\Pi^{*,c} := \left\{ \mathbb{P} \in \Pi^c \left| \mathbb{P}-\liminf_{t \to \infty} \frac{\log \eta^{*,c}(X_t)}{t} \geq 0 \mathbb{P}-\text{a.s.} \right. \right\}.
\]

Then we have

- \( \Pi^{*,c} \) includes all the measures in \( \Pi^c \) under which \( X \) is stable.
- \( \lambda^{*,c}(E) = \sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^{*,c}} g(\pi; \mathbb{P}) = \inf_{\mathbb{P} \in \Pi^{*,c}} \sup_{\pi \in \mathcal{V}} g(\pi; \mathbb{P}). \)
- \( \pi^{*,c}_t := e^{\lambda^{*,c}(E)t} \nabla \eta^{*,c}(X_t) \in \mathcal{V} \) satisfies

\[
g(\pi^{*,c}; \mathbb{P}) \geq \lambda^{*,c}(E), \quad \forall \mathbb{P} \in \Pi^{*,c}.
\]
When $c \in C$ is NOT given...

What operator should we use?

- When $c \in C$ is fixed, we use $L_c(\eta)(x) = \frac{1}{2} \text{Tr}[c(x)D^2\eta(x)]$.
- When $c \in C$ is NOT fixed, the appropriate operator could be

$$F(x, D^2\eta(x)) := \frac{1}{2} \sup_{A \in A(\theta(x), \Theta(x))} \text{Tr}[A D^2\eta(x)],$$

where $A(\lambda, \Lambda)$ denotes the set of matrices in $S^d$ with eigenvalues lying in $[\lambda, \Lambda]$.

The operator $F$ is a variant of Pucci’s extremal operator

$$\mathcal{M}^+_{\lambda, \Lambda}(M) := \sup_{A \in A(\lambda, \Lambda)} \text{Tr}(AM), \quad \forall \ M \in S^d,$$

where $0 < \lambda \leq \Lambda$ are some fixed constants.
When $c \in \mathcal{C}$ is NOT given...

For any $D \subset E$ and $\lambda \in \mathbb{R}$, we consider

$$H_\lambda(D) := \{ \eta \in C^2(D) \mid F(x, D^2\eta) + \lambda \eta \leq 0, \; \eta > 0 \text{ in } D \},$$

and define the principal eigenvalue for $F$ on $D$ as

$$\lambda^*(D) := \sup\{ \lambda \in \mathbb{R} \mid H_\lambda(D) \neq \emptyset \}.$$
When $c \in C$ is NOT given...

"arguments in KR12" + "$\lambda^*(E) = \inf_{c \in C} \lambda^{*,c}(E)$" =

**Theorem**

Take $\eta^* \in H_{\lambda^*(E)}(E)$. Define

$$\pi^*_t := e^{\lambda^*(E)t} \nabla \eta^*(X_t) \quad \forall \ t \geq 0,$$

and set

$$\Pi^* := \left\{ \mathbb{P} \in \Pi \mid \mathbb{P}-\lim_{t \to \infty} \inf_{t \geq 0} \mathbb{P}-a.s. \log \eta^*(X_t) \geq 0 \right\}.$$

Then, we have

- $\Pi^*$ includes all the measures in $\Pi$ under which $X$ is stable.
- $\lambda^*(E) = \sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(\pi; \mathbb{P}) = \inf_{\mathbb{P} \in \Pi^*} \sup_{\pi \in \mathcal{V}} g(\pi; \mathbb{P}).$
- $\pi^* \in \mathcal{V}$ and $g(\pi^*; \mathbb{P}) \geq \lambda^*(E)$ for all $\mathbb{P} \in \Pi^*$. 
Assume: there exist \( \{E_n\}_{n \in \mathbb{N}} \) of bounded open convex subsets of \( E \) s.t. \( \partial E_n \) is of \( C^{2,\alpha} \), \( \overline{E}_n \subset E_{n+1} \ \forall \ n \in \mathbb{N} \), and \( E = \bigcup_{n=1}^{\infty} E_n \).
Sketch of proof:

1. On each $E_n$, find a positive viscosity solution $\eta_n$ (using Quaas & Sirakov (2008)) to

\[ F(x, D^2\eta_n) + \lambda^*(E_n)\eta_n \leq 0. \tag{2} \]

2. Show that $\eta_n$ is actually smooth (using Safonov (1988)).

3. Show $\lambda^*(E_n) = \inf_{c \in C} \lambda^{*,c}(E_n)$.

   (I) \leq: Use a maximum principle related to $F$.

   (II) \geq: Find $\{c_m\}_{m \in \mathbb{N}}$ of measurable functions satisfying (1) s.t.

   \[ \lambda^*(E_n) \geq \liminf_{m \to \infty} \lambda^{*,c_m}(E_n). \tag{3} \]

Use “continuous selection” in Brown (1989) to construct $\{c'_m\}_{m \in \mathbb{N}}$ of continuous functions satisfying (1) s.t. (3) holds (smoothness of $\eta_n$ needed). Mollifying $\{c'_m\}_{m \in \mathbb{N}}$, get $\{c''_m\}_{m \in \mathbb{N}} \subset C$ s.t. (3) holds.
Proving \( \lambda^*(E) = \inf_{c \in C} \lambda^{*,c}(E) \)

4. Show \( \lambda^*(E) = \lambda_0 := \lim_{n \to \infty} \lambda^*(E_n) \).
   
   (I) \( \leq \): obvious from definitions.
   
   (II) \( \geq \): Prove a Harnack inequality for \( F \), which implies \( \eta_n \to \eta^* \) uniformly on \( E \). This and (2) yields

   \[ F(x, D^2 \eta^*) + \lambda_0 \eta^* \leq 0. \]

By Safonov (1988) again, \( \eta^* \) is smooth. Thus, conclude \( \eta^* \in H_{\lambda_0}(E) \), which gives \( \lambda^*(E) \geq \lambda_0 \).

5. Since \( \lambda^{*,c}(E) = \inf_{n \in \mathbb{N}} \lambda^{*,c}(E_n) \) for any \( BL \ c \in C \) (Pinsky (1995)),

   \[
   \inf_{c \in C} \lambda^{*,c}(E) = \inf_{c \in C} \inf_{n \in \mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n \in \mathbb{N}} \inf_{c \in C} \lambda^{*,c}(E_n) = \inf_{n \in \mathbb{N}} \lambda^*(E_n) = \lambda^*(E).
   \]
Among an appropriate class $\mathcal{C}$ of covariance structures, we characterize the **largest possible robust asymptotic growth rate** as the principle eigenvalue $\lambda^*(E)$ of the fully nonlinear elliptic operator $F$, and identify the **robust trading strategy** in terms of $\lambda^*(E)$ and **the associated eigenfunction**.

The covariance uncertainty we consider is similar to the **“Knightian uncertainty”** formulated in Fernholz & Karatzas (2011), in the sense that the constraint on covariance is Markovian. The latter, however, is more general as it allows **the covariance itself to be non-Markovian**. Can we generalize our results to the case with non-Markovian covariances?


Thank you very much for your attention!
Q & A