Dynamic Trading Volume

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- Geometric Brownian Motion: basic stochastic process for price. Since Samuelson, Black and Scholes, Merton.
- Basic stochastic process for volume?
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Unsettling Answers

- **Volume**: rate of change in total quantities traded.
- All these models: no frictions.
- Transaction costs, exogenous prices. After Constantinides (1986). Volume finite as time-average. Either zero (no trade region) or infinite (trading boundaries).
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This Talk

• **Question:** if price is geometric Brownian Motion, what is the process for volume?

• **Inputs**
  - Price exogenous. Geometric Brownian Motion.
  - Representative agent.
    - Constant relative risk aversion and long horizon.
  - Friction. Execution price linear in volume.

• **Outputs**
  - Stochastic process for trading volume.
  - Optimal trading policy and welfare.
  - Small friction asymptotics explicit.
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Market

- Brownian Motion \((W_t)_{t \geq 0}\) with natural filtration \((\mathcal{F}_t)_{t \geq 0}\).

- Best quoted price of risky asset. Price for an infinitesimal trade.

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

- Trade \(\Delta \theta\) shares over time interval \(\Delta t\). Order filled at price

\[
\tilde{S}_t(\Delta \theta) := S_t \left(1 + \lambda \frac{S_t \Delta \theta}{X_t \Delta t}\right)
\]

where \(X_t\) is investor’s wealth.

- \(\lambda\) measures illiquidity. \(1/\lambda\) market depth. Like Kyle’s (1985) lambda.

- Price worse for larger quantity \(|\Delta \theta|\) or shorter execution time \(\Delta t\). Price linear in quantity, inversely proportional to execution time.

- Same amount \(S_t \Delta \theta\) has lower impact if investor’s wealth larger.

- Makes model scale-invariant. Doubling wealth, and all subsequent trades, doubles final payoff exactly.
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Alternatives?

- Alternatives: quantities $\Delta \theta$, or share turnover $\Delta \theta / \theta$. Consequences?
  - Quantities ($\Delta \theta$):
    \[
    \tilde{S}_t(\Delta \theta) := S_t + \lambda \frac{\Delta \theta}{\Delta t}
    \]
  - Price impact independent of price. Not invariant to stock splits!
  - Suitable for short horizons (liquidation) or mean-variance criteria.
  - Share turnover:
    Stationary measure of trading volume (Lo and Wang, 2000). Observable.
    \[
    \tilde{S}_t(\Delta \theta) := S_t \left( 1 + \lambda \frac{\Delta \theta}{\theta_t \Delta t} \right)
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  - Problematic. Infinite price impact with cash position.
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Wealth and Portfolio

- Continuous trading: execution price $\tilde{S}_t(\dot{\theta}_t) = S_t \left(1 + \lambda \frac{\dot{\theta}_t S_t}{X_t}\right)$, cash position

$$dC_t = -S_t \left(1 + \lambda \frac{\dot{\theta}_t S_t}{X_t}\right) d\theta_t = -S_t \left(\dot{\theta}_t + \lambda \frac{S_t}{X_t} \dot{\theta}_t^2\right) dt$$

- Trading volume as wealth turnover $u_t := \frac{\dot{\theta}_t S_t}{X_t}$. Amount traded in unit of time, as fraction of wealth.

- Dynamics for wealth $X_t := \theta_t S_t + C_t$ and risky portfolio weight $Y_t := \frac{\theta_t S_t}{X_t}$

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- Illiquidity...

- ...reduces portfolio return $(-\lambda u_t^2)$.

  Turnover effect quadratic: quantities times price impact.

- ...increases risky weight $(\lambda Y_t u_t^2)$. Buy: pay more cash. Sell: get less cash.

  Turnover effect linear in risky weight $Y_t$. Vanishes for cash position.
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Admissible strategy: process \((u_t)_{t \geq 0}\), adapted to \(\mathcal{F}_t\), such that system

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- Contrast to models without frictions or with transaction costs: control variable is not risky weight \(Y_t\), but its “rate of change” \(u_t\).
- Portfolio weight \(Y_t\) is now a state variable.
- Illiquid vs. perfectly liquid market.
  Steering a ship vs. driving a race car.
- Frictionless solution \(Y_t = \frac{\mu}{\gamma \sigma^2}\) unfeasible. A still ship in stormy sea.
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- Contrast to models without frictions or with transaction costs: control variable is not risky weight \(Y_t\), but its “rate of change” \(u_t\).
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- Illiquid vs. perfectly liquid market.
  Steering a ship vs. driving a race car.
- Frictionless solution \(Y_t = \frac{\mu}{\gamma} \sigma^2\) unfeasible. A still ship in stormy sea.
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Definition

Admissible strategy: process \((u_t)_{t \geq 0}\), adapted to \(\mathcal{F}_t\), such that system

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- **Investor with relative risk aversion** $\gamma$.
- Maximize equivalent safe rate, i.e., power utility over long horizon:

$$\max_u \lim_{T \to \infty} \frac{1}{T} \log E \left[ X_T^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$

- Tradeoff between speed and impact.
- Optimal policy and welfare.
- Implied trading volume.
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Verifying

Theorem

If \( \frac{\mu}{\gamma \sigma^2} \in (0, 1) \), then the optimal wealth turnover and equivalent safe rate are:

\[
\hat{u}(y) = \frac{1}{2\lambda} \frac{q(y)}{1 - yq(y)} \quad \text{EsR}_\gamma(\hat{u}) = \beta
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where \( \beta \in (0, \frac{\mu^2}{2\gamma \sigma^2}) \) and \( q : [0, 1] \mapsto \mathbb{R} \) are the unique pair that solves the ODE

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-\beta + \mu y - \gamma \sigma^2 y^2 + y(1 - y)(\mu - \gamma \sigma^2 y)q + \frac{q^2}{4\lambda(1 - yq)} + \frac{\sigma^2}{2} y^2(1 - y)^2(q' + (1 - \gamma)q^2) = 0
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and \( q(0) = 2\sqrt{\lambda \beta}, \ q(1) = \lambda d - \sqrt{\lambda d(\lambda d - 2)} \), where \( d = -\gamma \sigma^2 - 2\beta + 2\mu \).

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\( \bar{Y} = \frac{\mu}{\gamma \sigma^2} \in (0, 1) \). Asymptotic expansions for turnover and equivalent safe rate:

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Long-term average of (unsigned) turnover

\[
|\operatorname{ET}| = \lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{u}(Y_t)| \, dt = \pi^{-1/2} \sigma^{3/2} \bar{Y} (1 - \bar{Y}) \left( \gamma / 2 \right)^{1/4} \lambda^{-1/4} + O(\lambda^{1/2})
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- Trade faster with more volatility. Volume typically increases with volatility.
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Turnover ($\mu = 8\%, \sigma = 16\%, \lambda = 10^{-3}, \gamma = 5$)

- Solution to ODE almost linear. Asymptotics accurate.
Turnover ($\mu = 8\%$, $\sigma = 16\%$, $\lambda = 10^{-3}$, $\gamma = 10$)

- Higher risk aversion: lower target and narrower interval.
Effect of Illiquidity ($\mu = 8\%$, $\sigma = 16\%$, $\gamma = 5$)

- $\lambda = 10^{-4}$ (solid), $10^{-3}$ (long), $10^{-2}$, (short), and $10^{-1}$ (dotted).
Trading Volume

- Wealth turnover approximately Ornstein-Uhlenbeck:

\[
\text{\(d\hat{u}_t = \sigma \sqrt{\frac{\gamma}{2\lambda}} (\sigma^2 \bar{Y}^2 (1 - \bar{Y})(1 - \gamma) - \hat{u}_t) \, dt - \sigma^2 \sqrt{\frac{\gamma}{2\lambda}} \bar{Y} (1 - \bar{Y}) \, dW_t\)}
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- In the following sense:

Theorem

The process \(\hat{u}_t\) has asymptotic moments:

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How big is $\lambda$?

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$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \frac{u(Y_t)}{Y_t} \right| dt = \pi^{-1/2} \sigma^{3/2} (1 - \bar{Y}) (\gamma/2)^{1/4} \lambda^{-1/4}$$

- Match formula with observed share turnover. Bounds on $\gamma$, $\bar{Y}$.

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<th>Period</th>
<th>Volatility</th>
<th>Share Turnover</th>
<th>$-\log_{10} \lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>low</td>
</tr>
<tr>
<td>1926-1929</td>
<td>20%</td>
<td>125%</td>
<td>6.164</td>
</tr>
<tr>
<td>1930-1939</td>
<td>30%</td>
<td>39%</td>
<td>3.082</td>
</tr>
<tr>
<td>1940-1949</td>
<td>14%</td>
<td>12%</td>
<td>3.085</td>
</tr>
<tr>
<td>1950-1959</td>
<td>10%</td>
<td>12%</td>
<td>3.855</td>
</tr>
<tr>
<td>1960-1969</td>
<td>10%</td>
<td>15%</td>
<td>4.365</td>
</tr>
<tr>
<td>1970-1979</td>
<td>13%</td>
<td>20%</td>
<td>4.107</td>
</tr>
<tr>
<td>1980-1989</td>
<td>15%</td>
<td>63%</td>
<td>5.699</td>
</tr>
<tr>
<td>1990-1999</td>
<td>13%</td>
<td>95%</td>
<td>6.821</td>
</tr>
<tr>
<td>2000-2009</td>
<td>22%</td>
<td>199%</td>
<td>6.708</td>
</tr>
</tbody>
</table>

- $\gamma = 1$, $\bar{Y} = 1/2$ (high), and $\gamma = 10$, $\bar{Y} = 0$ (low).
Welfare

• Define welfare loss as decrease in equivalent safe rate due to friction:

\[ \text{LoS} = \frac{\mu^2}{2\gamma\sigma^2} - \text{EsR}_{\gamma}(\hat{u}) \approx \sigma^3 \sqrt{\frac{\gamma}{2}} \bar{Y}^2 (1 - \bar{Y})^2 \lambda^{1/2} \]

• Zero loss if no trading necessary, i.e. \( \bar{Y} \in \{0, 1\} \).
• Universal relation:

\[ \text{LoS} = \pi \lambda |ET|^2 \]

Welfare loss equals squared turnover times liquidity, times \( \pi \).
• Compare to proportional transaction costs \( \varepsilon \):

\[ \text{LoS} = \frac{\mu^2}{2\gamma\sigma^2} - \text{EsR}_{\gamma} \approx \frac{\gamma\sigma^2}{2} \left( \frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{2/3} \varepsilon^{2/3} \]

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Welfare loss equals turnover times spread, times constant \( 3/4 \).
• Linear effect with transaction costs (price, not quantity). Quadratic effect with liquidity (price \textit{times} quantity).
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#### Theorem

If \( \frac{\mu}{\gamma \sigma^2} \leq 0 \), then \( Y_t = 0 \) and \( \hat{u} = 0 \) for all \( t \) optimal. Equivalent safe rate zero.

If \( \frac{\mu}{\gamma \sigma^2} \geq 1 \), then \( Y_t = 1 \) and \( \hat{u} = 0 \) for all \( t \) optimal. Equivalent safe rate \( \mu - \frac{\gamma}{2} \sigma^2 \).

- If Merton investor shorts, keep all wealth in safe asset, but do not short.
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Control Argument

- Value function $v$ depends on (1) current wealth $X_t$, (2) current risky weight $Y_t$, and (3) calendar time $t$.

- Evolution for fixed trading strategy $u = \frac{\dot{\theta}_t S_t}{X_t}$:

  \[
  dv(X_t, Y_t, t) = v_t dt + v_x (\mu X_t Y_t - \lambda X_t u_t^2) dt + v_x X_t Y_t \sigma dW_t \\
  + v_y (Y_t (1 - Y_t) (\mu - Y_t \sigma^2) + u_t + \lambda Y_t u_t^2) dt + v_y Y_t (1 - Y_t) \sigma dW_t \\
  + \left( \frac{\sigma^2}{2} v_{xx} X_t^2 Y_t^2 + \frac{\sigma^2}{2} v_{yy} Y_t^2 (1 - Y_t)^2 + \sigma^2 v_{xy} X_t Y_t^2 (1 - Y_t) \right) dt
  \]

- Maximize drift over $u$, and set result equal to zero:

  \[
  v_t + \max_u \left( v_x (\mu x y - \lambda x u^2) + v_y (y (1 - y) (\mu - \sigma^2 y) + u + \lambda y u^2) \\
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Homogeneity and Long-Run

- Homogeneity in wealth $v(t, x, y) = x^{1-\gamma} v(t, 1, y)$.
- Guess long-term growth at equivalent safe rate $\beta$, to be found.
- Substitution $v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)(\beta(T-t)+\int y q(z) dz)}$ reduces HJB equation

$$-\beta + \max_u \left( \left( \mu y - \gamma \frac{\sigma^2}{2} y^2 - \lambda u^2 \right) + q(y(1-y)(\mu - \gamma \sigma^2 y) + u + \lambda y u^2 \right) + \frac{\sigma^2}{2} y^2 (1-y)^2 \left( q' + (1-\gamma) q^2 \right) \right) = 0.$$  

- Maximum for $u(y) = \frac{q(y)}{2\lambda(1-y q(y))}$.
- Plugging yields

$$\mu y - \gamma \frac{\sigma^2}{2} y^2 + y(1-y)(\mu - \gamma \sigma^2 y) q + \frac{q^2}{4\lambda(1-y q)} + \frac{\sigma^2}{2} y^2 (1-y)^2 (q' + (1-\gamma) q^2) = \beta$$

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- Homogeneity in wealth $\nu(t, x, y) = x^{1-\gamma} \nu(t, 1, y)$.
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\[-\beta + \max_u \left( \left( \mu y - \frac{\gamma \sigma^2}{2} y^2 - \lambda u^2 \right) + q(y(1-y)(\mu - \gamma \sigma^2 y) + u + \lambda yu^2 \right) + \frac{\sigma^2}{2} y^2 (1-y)^2 (q' + (1-\gamma)q^2) \right) = 0.\]

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- Expand equivalent safe rate as \( \beta = \frac{\mu^2}{2\gamma\sigma^2} - c(\lambda) \)
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Verification

Lemma

Let $q$ solve the HJB equation, and define $Q(y) = \int_{y}^{\infty} q(z)dz$. There exists a probability $\hat{P}$, equivalent to $P$, such that the terminal wealth $X_T$ of any admissible strategy satisfies:

$$E[X_T^{1-\gamma}]^{\frac{1}{1-\gamma}} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{\frac{1}{1-\gamma}},$$

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any $\beta$, for corresponding $Q$.
- Idea: pick largest $\beta^*$ to make $Q$ disappear in the long run.
- A priori bounds:
  $$\beta^* < \frac{\mu^2}{2\gamma \sigma^2}$$
  (frictionless solution)
  $$\max \left(0, \mu - \frac{\gamma}{2} \sigma^2\right) < \beta^*$$
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**Existence**

**Theorem**

Assume $0 < \frac{\mu}{\gamma \sigma^2} < 1$. There exists $\beta^*$ such that HJB equation has solution $q(y)$ with positive finite limit in 0 and negative finite limit in 1.

- for $\beta > 0$, there exists a unique solution $q_{0,\beta}(y)$ to HJB equation with positive finite limit in 0 (and the limit is $2\sqrt{\lambda \beta}$);
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- there exists $\beta_u$ such that $q_{0, \beta_u}(y) > q_{1, \beta_u}(y)$ for some $y$;
- there exists $\beta_l$ such that $q_{0, \beta_l}(y) < q_{1, \beta_l}(y)$ for some $y$;
- by continuity and boundedness, there exists $\beta^* \in (\beta_l, \beta_u)$ such that $q_{0, \beta^*}(y) = q_{1, \beta^*}(y)$.
- Boundary conditions are natural!
Explosion with Leverage

**Theorem**

If $Y_t$ that satisfies $Y_0 \in (1, +\infty)$ and

$$dY_t = Y_t(1 - Y_t)(\mu dt - Y_t \sigma^2 dt + \sigma dW_t) + u_t(1 + \lambda Y_t u_t)dt$$

explodes in finite time with positive probability.

- For $y \in [1, +\infty)$, drift bounded below by $\tilde{\mu}(y) := y(1 - y)(\mu - y \sigma^2) - \frac{1}{4\lambda}$.
- Comparison principle: enough to check explosion with lower bound drift.
- Scale function $s(x) = \int_c^x \exp \left( -2 \int_c^y \frac{\tilde{\mu}(z)}{\sigma^2(z)} dz \right) dy$ finite at both 1 and $\infty$.
- For $y \sim 1$, $\tilde{\mu}(y) \sim -\frac{1}{4\lambda}$ and $\tilde{\sigma}^2(y) \sim \sigma^2(1 - y)^2$.
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Bankruptcy

• τ explosion time for $Y_t$. Show that $X_\tau(\omega) = 0$ on $\omega \in \{\tau < +\infty\}$.

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$$X_T = xe^{\int_0^T (Y_t\mu - \lambda u_t^2 - \frac{\sigma^2}{2} Y_t^2) dt + \sigma \int_0^T Y_t dW_t}.$$  

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Optimality

- Check that $Y_t \equiv 1$ optimal if $\frac{\mu}{\gamma \sigma^2} > 1$.
- By Itô's formula,
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  X_T^{1-\gamma} = x^{1-\gamma} e^{(1-\gamma) \int_0^T (Y_t \mu - \frac{1}{2} Y_t^2 \sigma^2 - \lambda u_t^2) dt + (1-\gamma) \int_0^T Y_t \sigma dW_t}.
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- and hence, for $g(y, u) = y\mu - \frac{1}{2} y^2 \gamma \sigma^2 - \lambda u^2$,
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- and hence, for $g(y, u) = y \mu - \frac{1}{2} y^2 \gamma \sigma^2 - \lambda u^2$,
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  where $\frac{d\hat{P}}{dP} = \exp \{ \int_0^T -\frac{1}{2} Y_t^2 (1-\gamma)^2 \sigma^2 dt + \int_0^T Y_t (1-\gamma) \sigma dW_t \}$.
- $g(y, u)$ on $[0, 1] \times \mathbb{R}$ has maximum $g(1, 0) = \mu - \frac{1}{2} \gamma \sigma^2$ at $(1, 0)$.
- Since $Y_t \equiv 1$ and $u_t \equiv 0$ is admissible, it is also optimal.
Optimality

- Check that \( Y_t \equiv 1 \) optimal if \( \frac{\mu}{\gamma \sigma^2} > 1 \).
- By Itô’s formula,

\[
X_T^{1-\gamma} = x^{1-\gamma} e^{(1-\gamma) \int_0^T (Y_t \mu - \frac{1}{2} Y_t^2 \sigma^2 - \lambda u_t^2) dt + (1-\gamma) \int_0^T Y_t \sigma dW_t}.
\]

- and hence, for \( g(y, u) = y \mu - \frac{1}{2} y^2 \gamma \sigma^2 - \lambda u^2 \),

\[
E[X_T^{1-\gamma}] = x^{1-\gamma} E_{\tilde{P}} \left[ e^{\int_0^T (1-\gamma) g(Y_t, u_t) dt} \right],
\]

where \( \frac{d\tilde{P}}{dP} = \exp \{ \int_0^T -\frac{1}{2} Y_t^2 (1 - \gamma)^2 \sigma^2 dt + \int_0^T Y_t (1 - \gamma) \sigma dW_t \} \).

- \( g(y, u) \) on \([0, 1] \times \mathbb{R}\) has maximum \( g(1, 0) = \mu - \frac{1}{2} \gamma \sigma^2 \) at \((1, 0)\).
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- Finite market depth. Execution price linear in volume as wealth turnover.
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- Base price geometric Brownian Motion.
- Trade towards frictionless portfolio.
- Dynamics for trading volume.
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ERC Starting Grant

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