Pattern formation in compressed elastic films on compliant substrates: an explanation of the herringbone structure

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Joint work with Hoai-Minh Nguyen

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The elastic energy of a thin sheet is like a Landau theory:

\[
E = (\text{membrane energy}) + h^2(\text{bending energy})
\]

= nonconvex term + regularizing singular perturbation

In phase transformation: nonconvexity \(\Rightarrow\) microstructure; surface energy \(\Rightarrow\) length scale and pattern. Examples:

- **martensitic transformation** – martensite twins near an austenite interface (Kohn & Müller)
- **uniaxial ferromagnets** – refinement of domains near the basal plane (Choksi, Kohn, Otto)
- **type-I superconductors** – microstructure of the intermediate state (Choksi, Conti, Kohn, Otto; Knüpfer & Muratov)

**Current thrust**: an analogous understanding of wrinkling. **Motivations:**

- **Provide new insight** in a setting with plenty of physical interest, where theory and experiment meet.
- **Develop new tools** in the calculus of variations. Bending energy is different from surface energy!
The big picture – continued

Some current projects concerning wrinkling as microstructure

Planar annulus loaded in tension (Bella & Kohn, preprint). Motivated by wrinkling around a drop on a floating sheet.

Hanging drapes (Bella & Kohn, in progress). Energy-based explanation of this familiar phenomenon.

Floating sheet, confined on two sides (Nguyen & Kohn, focus of Hoai-Minh’s talk). Wrinkles refine near the free edge; why?

Compressed thin film bonded to a compliant substrate (Nguyen & Kohn, focus of this talk).
Some closely related themes

Blisters formed by compressed thin films (Jin & Sternberg; Ben Belgacem, Conti, DeSimone, & Müller; also Bedrossian and Kohn, in progress)

Conical singularities (dcones) (Brandman, Nguyen, & Kohn – focus of Jeremy’s talk; Olbermann & Müller – focus of Heiner’s talk; much physics literature, cf review by Witten).

Ridges and crumpling (Venkataramani, Conti & Maggi; much physics literature, cf review by Witten).
Today’s topic

Wrinkling of a compressed thin film bonded to a thick, compliant substrate:

1. Phenomenology: herringbones and labyrinths
2. Energy minimization (using von Karman theory)
3. The energy scaling law (herringbone pattern is optimal)
4. Sketch of the upper bound (the herringbone pattern)
5. Sketch of the lower bound (no other pattern does better)
6. Conclusions (achievements and open problems)

Joint with Hoai-Minh Nguyen
Preprint at math.cims.nyu.edu/faculty/kohn
Wrinkling of thin films compressed by thick, compliant substrates:

- deposit film at high temp then cool; or
- deposit on stretched substrate then release;
- film buckles to avoid compression

Commonly seen pattern: herringbone

silicon on pdms

gold on pdms
Herringbone pattern when film has some anisotropy, or for specific release histories. Otherwise a less ordered “labyrinth” pattern.

- silicon on pdms

- gold on pdms

- different release histories
The elastic energy

We use a “small-slope” (von Karman) version of elasticity, writing \((w_1, w_2, u_3)\) for the elastic displacement. The energy has three terms:

1. **Membrane energy** captures fact that film’s natural length is larger than that of the substrate:

\[
\alpha_m h \int |e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta l|^2 \, dx \, dy
\]

2. **Bending energy** captures resistance to bending:

\[
h^3 \int |\nabla \nabla u_3|^2 \, dx \, dy
\]

3. **Substrate energy** captures fact that substrate acts as a “spring”, tending to keep film flat:

\[
\alpha_s \left( \|w\|_{H^{1/2}}^2 + \|u_3\|_{H^{1/2}}^2 \right)
\]

where \(\|g\|_{H^{1/2}}^2 = \sum |k| |\hat{g}(k)|^2\)
The membrane energy

\[ E_{\text{membrane}} = \alpha_m h \int \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy \]

where \((w_1, w_2, u_3)\) is the elastic displacement, and \(\eta\) is the misfit.

- 1D analogue: \( \int |\partial_x w_1 + \frac{1}{2} (\partial_x u_3)^2 - \eta|^2 \, dx \)
- Explanation: if \((x, 0) \mapsto (x + w_1(x), u_3(x))\) then local stretching is
  \[
  \sqrt{(1 + \partial_x w_1)^2 + (\partial_x u_3)^2} - 1 \approx \partial_x w_1 + \frac{1}{2} (\partial_x u_3)^2
  \]
- Vanishes in 1D for sinusoidal profile:
  \[ w_1 = \eta \frac{\lambda}{4} \sin(4x/\lambda), \quad u_3 = \sqrt{\eta \lambda} \cos(2x/\lambda) \]
- Our problem is 2D, with isotropic misfit \(\eta I\);
- The herringbone pattern uses sinusoidal wrinkling in two distinct orientations. It does better than the Miura-ori pattern.
The bending energy

\[ E_{\text{bending}} = h^3 \int |\nabla \nabla u_3|^2 \, dx \, dy \]

where \( u_3 \) is the out-of-plane displacement.

- In a **nonlinear theory**, bending energy \( \sim h^3 \int \kappa_1^2 + \kappa_2^2 \) where \( \kappa_i \) are the principal curvatures.
- In our **small-slope** (von-Karman) setting, the principal curvatures are the eigenvalues of \( \nabla \nabla u_3 \).
The substrate energy

\[ E_{\text{substrate}} = \alpha_s \left( \| w \|_{H^{1/2}}^2 + \| u_3 \|_{H^{1/2}}^2 \right) \]

where \((w_1, w_2, u_3)\) is the (periodic) elastic displacement, and
\[ \| g \|_{H^{1/2}}^2 = \sum |k| \| \hat{g}(k) \|^2. \]

- Treat substrate as **semi-infinite** isotropic elastic halfspace.
- Given surface displacement \((w_1, w_2, u_3)\), solve 3D linear elasticity problem in substrate by separation of variables.
- Substrate energy is the result (modulo constants).
Total energy = membrane + bending + substrate

To permit spatial averaging, we assume periodicity on some (large) scale $L$, and we focus on the energy per unit area:

$$E_h = \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy \quad \text{(membrane)}$$

$$+ \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla \nabla u_3|^2 \, dx \, dy \quad \text{(bending)}$$

$$+ \frac{\alpha_s}{L^2} \left( \| w \|_{H^{1/2}}^2 + \| u_3 \|_{H^{1/2}}^2 \right) \quad \text{(substrate)}$$

where $h$ is the thickness of the film.

- We have already normalized by Young’s modulus of the film, so $\alpha_m, \alpha_s, \eta$ are dimensionless parameters:
  - $\alpha_m$ (order 1) comes from justin of von Karman theory;
  - $\alpha_s$ (small) is the ratio (substrate stiffness)/(film stiffness);
  - $\eta$ (small) is the misfit.

- Unwrinkled state $(w_1, w_2, u_3) = 0$ has energy $\alpha_m \eta^2 h$.

- Bending term has factor $h^3$ while membrane term has factor $h$. 
The energy scaling law

**Theorem**

If $h/L$ and $\eta$ are small enough, the minimum energy satisfies

$$\min E_h \sim \min \{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\};$$

moreover

- the first alternative corresponds to the **unbuckled state**; it is better when $\alpha_m \eta < \alpha_s^{2/3}$.

- the second alternative is achieved by a **herringbone pattern** using wrinkles with length scale $\lambda = \alpha_s^{-1/3} h$, whose direction oscillates on a suitable length scale (not fully determined).

The smallness conditions are explicit:

$$\alpha_m \alpha_s^{-4/3} (h/L)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_s^{2/3}.$$

Perhaps other, less-ordered patterns could also be optimal (e.g. “labyrinths”). Numerical results suggest this is the case.
The energy scaling law – cont’d

\[ \min E_h \sim \min \{ \alpha_m \eta^2 h, \alpha_s^{2/3} \eta h \}; \]

One consequence: the Miura-ori pattern is not optimal:

- Its scaling law is \( \alpha_m^{1/6} \alpha_s^{5/8} \eta^{17/16} h \).
- If film prefers to buckle (\( \alpha_m \eta \gg \alpha_s^{2/3} \)) then Miura-ori energy \( \gg \) herringbone energy.

Intuition:

- Bending energy requires folds of Miura-ori pattern to be rounded.
- Where folds intersect this costs significant membrane energy.
- In herringbone pattern the membrane term isn’t identically zero, but it does not contribute at leading order.
The herringbone pattern

- Mixture of two symmetry-related “phases”
- “Phase 1” uses sinusoidal wrinkles perp to \((1, 1)\), superimposed on an in-plane shear.
- “Phase 2” uses wrinkles perp to \((1, -1)\), superimposed on a different shear.

In phase 1: 
\[
e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} \eta & \eta \\ \eta & \eta \end{pmatrix} + \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} = \eta I;
\]

In phase 2: 
\[
e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} \eta & -\eta \\ -\eta & \eta \end{pmatrix} + \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} = \eta I;
\]

Membrane term vanishes! Since the average in-plane shear is 0, the in-plane displacement \(w\) can be periodic.

Herringbone pattern has **two length scales**: 
- The smaller one (the scale of the wrinkling) is set by competition between *bending* term and *substrate energy of \(u_3\).*
- The larger one (scale of the phase mixture) must be s.t. the *substrate energy of \(w\) is insignificant.* (It is not fully determined.)
Phenomenology - review

silicon on pdms

gold on pdms

different release histories
Claim: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

Let’s take \(L = 1\) for simplicity. We’ll use only that

\[
\text{membrane term} \geq \alpha_m h \int |\partial_x w_1 + \frac{1}{2} |\partial_x u_3|^2 - \eta|^2 \, dx \, dy,
\]

\[
\text{bending term} = h^3 \|\nabla \nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.
\]

**CASE 1:** If \(\int |\nabla u_3|^2 \ll \eta\) then stretching \(\gtrsim \alpha_m \eta^2 h\), since \(\partial_x w_1\) has mean 0.

**CASE 2:** If \(\int |\nabla u_3|^2 \gtrsim \eta\) use the interpolation inequality

\[
\|\nabla u_3\|_{L^2} \lesssim \|\nabla \nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}
\]

to see that

\[
\text{Bending + substrate terms} \geq h^3 \|\nabla \nabla u_3\|_{L^2}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 \gtrsim \left( h^3 \|\nabla \nabla u_3\|_{L^2}^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^2 \right)^{1/3}
\]

\[\gtrsim h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta\]

using arith mean/geom mean inequality.
No other pattern can do better

Claim: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

Let’s take \(L = 1\) for simplicity. We’ll use only that

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\text{membrane term} \geq \alpha_m h \int |\partial_x w_1 + \frac{1}{2} |\partial_x u_3|^2 - \eta|^2 \, dx \, dy,
\]

\[
\text{bending term} = h^3 \|\nabla \nabla u_3\|_{L^2}^2,
\]

\[
\text{and substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.
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**CASE 1:** If \(\int |\nabla u_3|^2 \ll \eta\) then stretching \(\gtrsim \alpha_m \eta^2 h\), since \(\partial_x w_1\) has mean 0.

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to see that

\[
\text{Bending + substrate terms} = h^3 \|\nabla \nabla u_3\|_{L^2}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 \\
\gtrsim \left(h^3 \|\nabla \nabla u_3\|_{L^2}^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4 \right)^{1/3} \\
\gtrsim h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta
\]

using arith mean/geom mean inequality.
Claim: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

Let’s take \(L = 1\) for simplicity. We’ll use only that

\[
\text{membrane term} \geq \alpha_m h \int |\partial_x w_1 + \frac{1}{2} |\partial_x u_3|^2 - \eta|^2 \, dx \, dy,
\]

bending term \(= h^3 \|\nabla \nabla u_3\|_{L^2}^2\), and substrate term \(\geq \alpha_s \|u_3\|_{H^{1/2}}^2\).

CASE 1: If \(\int |\nabla u_3|^2 \ll \eta\) then stretching \(\gtrsim \alpha_m \eta^2 h\), since \(\partial_x w_1\) has mean 0.

CASE 2: If \(\int |\nabla u_3|^2 \gtrsim \eta\) use the interpolation inequality

\[
\|\nabla u_3\|_{L^2} \lesssim \|\nabla \nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}
\]

to see that

\[
\text{Bending + substrate terms} = h^3 \|\nabla \nabla u_3\|^2 + \frac{1}{2} \alpha_s \|u_3\|^2_{H^{1/2}} + \frac{1}{2} \alpha_s \|u_3\|^4_{H^{1/2}} \\
\gtrsim \left(h^3 \|\nabla \nabla u_3\|^{2} \alpha_s^{2} \|u_3\|_{H^{1/2}}^{4}\right)^{1/3} \\
\gtrsim h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^{2} \gtrsim h \alpha_s^{2/3} \eta
\]

using arith mean/geom mean inequality.
Stepping back

Main accomplishment: scaling law of the minimum energy, based on
- upper bound, corresponding to the herringbone pattern, and
- lower bound, using nothing more than interpolation.

Key point: they have the same scaling law as $h/L \rightarrow 0$.

Open questions:
- When is the von Karman model adequate? What changes when the slope of the wrinkling gets large, and/or the strain in the substrate becomes large?
- Can less-ordered patterns achieve the same scaling law?

State of the art: we’re developing tools by doing examples.
Sometimes tensile effects determine the direction of wrinkling, and the subtle task is to understand its length scale. Examples: wrinkling at the edge of a confined floating sheet (Hoai-Minh Nguyen’s talk), wrinkling of a hanging drape (Peter Bella and RVK, in preparation), and wrinkling near a drop on a floating sheet (expt: Huang et al).

Other times the loads are compressive and the geometry has much more freedom. Examples: crumpling of paper, blisters in compressed thin films, and our herringbones.


Website of Chris Santangelo, blogs.umass.edu, see also J. Huang et al, *Phys Rev Lett* 105 (2010) 038302


