Sign Patterns That Allow Strong Eventual Nonnegativity

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Sign Patterns

Definition

▶ A sign pattern matrix (or sign pattern) $A$, is a matrix having entries in $\{+, -, 0\}$.

▶ For a real matrix $A$, $\text{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in $A$.

▶ The set $Q(A)$ of all matrices $A \in \mathbb{R}^{n \times n}$ such that $\text{sgn}(A) = A$ is called the qualitative class of $A$.

▶ A real matrix $A \in Q(A)$ is a realization of $A$. 
The Digraph of a Sign Pattern

Definition
If $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ sign pattern, then the digraph of $\mathcal{A}$ is

$$\Gamma(\mathcal{A}) = (\{1, \ldots, n\}, \{(i, j) : \alpha_{ij} \neq 0\}).$$

Definition
A digraph is strongly connected if for any two vertices $u$ and $v$, there is a path from $u$ to $v$. 
Definition
A matrix $A$ is \textit{reducible} if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} B & 0 \\ X & C \end{bmatrix}$$

where $B$ and $C$ are square matrices, 0 is a zero matrix, and $X$ is any matrix. If such $P$ does not exist, then $A$ is \textit{irreducible}.

Fact
A matrix or a sign pattern is irreducible if and only if its digraph is strongly connected.
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A matrix $A$ is *reducible* if there exists a permutation matrix $P$ such that

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Fact
*A matrix or a sign pattern is irreducible if and only if its digraph is strongly connected.*
Definition
A digraph is *primitive* if:

1. It is strongly connected, and
2. the greatest common divisor of the cycle-lengths is 1.

Definition
A nonnegative sign pattern $A$ is *primitive* if its digraph is primitive.
Perron’s Theorem (1904)

If $A \in \mathbb{R}^{n \times n}$ is a (entry-wise) positive matrix (denoted $A > 0$), then

1. $\rho(A)$ is a simple (nonzero) eigenvalue.
2. $\rho(A)$ has positive right (and left) eigenvector.

Perron-Frobenius Theorem

Let $A$ be an irreducible nonnegative matrix. Then

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Definition
A matrix $A$ has the *strong Perron-Frobenius property* if $A$ has a simple real eigenvalue $\lambda$ where:

1. $\lambda = \rho(A)$.
2. If $\alpha$ is any other eigenvalue, then $|\alpha| < \lambda$.
3. The corresponding (right) eigenvector is positive.
EP Matrices

Definition
A matrix $A \in \mathbb{R}^{n \times n}$ is \textit{eventually positive} (EP) (\textit{eventually nonnegative}, EN) if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$ ($A^k \geq 0$).

Theorem (Handelman 1981)
Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

1. $A$ is EP.
2. Both $A$ and $A^T$ satisfy the strong Perron-Frobenius property.
3. There exists a $k \in \mathbb{Z}^+$ such that $A^k > 0$ and $A^{k+1} > 0$. 
EP Matrices

Definition
A matrix $A \in \mathbb{R}^{n \times n}$ is *eventually positive* (EP) (*eventually nonnegative*, EN) if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$ ($A^k \geq 0$).

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SEN Matrices

Definition
A matrix $A$ is *strongly eventually nonnegative (SEN)* if $A$ is EN and some power of $A$ is both nonnegative and irreducible.

Observation
If a matrix $A$ is EP, then $A$ is SEN.
SEN Matrices

Definition
A matrix $A$ is strongly eventually nonnegative (SEN) if $A$ is EN and some power of $A$ is both nonnegative and irreducible.

Observation
If a matrix $A$ is EP, then $A$ is SEN.
Example

The matrix

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -\frac{1}{2} \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

is SEN but not EP. Both \(A^2\) and \(A^3\) are nonnegative and \(A^3\) is irreducible. \(\Gamma(A)\) is bipartite, so \(A\) is not EP.
Example

The matrix

\[ A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

is SEN but not EP. Both \( A^2 \) and \( A^3 \) are nonnegative and \( A^3 \) is irreducible. \( \Gamma(A) \) is bipartite, so \( A \) is not EP.
Proposition

Let $A$ be an SEN matrix. Then $\rho(A)$ is a simple eigenvalue of $A$ having positive left and right eigenvectors. With the notation $\rho = \rho(A)$, $r = \#$ dominant eigenvalues of $A$, and $\omega = e^{2\pi i / r}$, the dominant eigenvalues of $A$ are 

$$\{\rho, \rho \omega, \ldots, \rho \omega^{r-1}\}.$$
Definition

For $r \geq 2$, matrix $A \in \mathbb{R}^{n \times n}$ is called $r$-cyclic if there exists a permutation matrix $P$ such that $PAP^T$ has the block form

$$
\begin{bmatrix}
0 & A_{12} & 0 & \ldots & 0 \\
0 & 0 & A_{23} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{r-1,r} \\
A_{r1} & 0 & 0 & \ldots & 0
\end{bmatrix}
$$

Observation

No $r$-cyclic matrix is EP.
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Observation
No \( r \)-cyclic matrix is EP.
Definition
A sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) (potentially eventually nonnegative, PEN) if there exists some realization $A \in Q(\mathcal{A})$ such that $A$ is EP (EN).

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Observation (Handelman, 1981)

If $A$ is a PEP sign pattern, then every row and column has at least one $+$ entry.

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If $A$ is a PSEN sign pattern, then every row and column has at least one $+$ entry.
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Observation

If $A$ is a PSEN sign pattern, then every row and column has at least one $+$ entry.
Subpatterns and Superpatterns

Definition
Let $\mathcal{A}$ and $\mathcal{B}$ be sign patterns. If you can obtain $\mathcal{B}$ by changing any of the entries in $\mathcal{A}$ to 0, then

- $\mathcal{B}$ is a subpattern of $\mathcal{A}$.
- $\mathcal{A}$ is a superpattern of $\mathcal{B}$.

Note that $\mathcal{A}$ is both a subpattern and superpattern of itself.

Example

Let $\mathcal{B} = \begin{bmatrix} - & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{A} = \begin{bmatrix} - & + \\ - & 0 \end{bmatrix}$.

Then $\mathcal{B}$ is a subpattern of $\mathcal{A}$, and $\mathcal{A}$ is a superpattern of $\mathcal{B}$. 
Subpatterns and Superpatterns

Definition
Let $A$ and $B$ be sign patterns. If you can obtain $B$ by changing any of the entries in $A$ to 0, then

- $B$ is a subpattern of $A$.
- $A$ is a superpattern of $B$.

Note that $A$ is both a subpattern and superpattern of itself.

Example
Let $B = \begin{bmatrix} - & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} - & + \\ - & 0 \end{bmatrix}$. Then $B$ is a subpattern of $A$, and $A$ is a superpattern of $B$. 
Subpatterns and Superpatterns

Theorem (AIM paper - 2010)

1. Let $A$ be a PEP sign pattern, then every superpattern of $A$ is PEP.

2. Let $B$ be a sign pattern which is not PEP, then no subpattern of $B$ is PEP.

The pattern $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ is PSEN; however, $\begin{bmatrix} - & + \\ + & 0 \end{bmatrix}$ is not PSEN since the dominant eigenvalue of any realization is negative.
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The pattern $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ is PSEN; however, $\begin{bmatrix} - & + \\ + & 0 \end{bmatrix}$ is not PSEN since the dominant eigenvalue of any realization is negative.
In the 2010 AIM paper it is shown that the minimum number of $+$ entries in a PEP sign pattern is $n + 1$.

The adjacency graph of the cycle on $n$ vertices is SEN: the minimum number of $+$ entries in a PSEN sign pattern is $n$.

**Theorem**

Let $A$ be an $n \times n$ PSEN sign pattern with exactly $n +$ entries. Then $A \geq 0$, i.e., $\Gamma(A)$ is the directed $n$-cycle.
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**Theorem**

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**Theorem**

Let $A$ be an $n \times n$ PSEN sign pattern with exactly $n + 1$ entries. Then $A \geq 0$, i.e., $\Gamma(A)$ is the directed $n$-cycle.
Theorem

If $A$ is PSEN, then $A$ is either PEP or $r$-cyclic.

Sketch of proof:

- Let $A \in Q(A)$ be SEN and $r$ be # of dominant eigenvalues.
- If $r = 1$, $A$ is EP so assume $r \geq 2$.
- $A^{kr+1} \geq 0$ is $r$-cyclic for $k > \max\{k_0, n\}$.
- Partition $A$ conformally with $A^{kr+1}$ (permute if necessary to get in “nice” form).
- If $A$ is not $r$-cyclic, there are some nonzero entries outside the $r$-cyclic components.
- Perturbing some of those entries will yield an EP matrix.
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- If $A$ is not $r$-cyclic, there are some nonzero entries outside the $r$-cyclic components.
- Perturbing some of those entries will yield an EP matrix.
Classification of small PSEN sign patterns

Definition
Two sign patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a permutation matrix $P$ such that $\mathcal{B} = P \mathcal{A} P^T$ or $\mathcal{B} = P \mathcal{A}^T P^T$. 

Theorem

For an $n \times n$ sign pattern $A$ with $n \leq 3$, $A$ is PSEN if and only if one of the following is true:

i. $A^+$ is primitive (hence $A$ is PEP).

ii. $n = 3$ and $A$ is equivalent to a sign pattern of the form

$$B = \begin{bmatrix}
+ & - & \ominus \\
+ & ? & - \\
- & + & +
\end{bmatrix}.$$  

Where $\ominus$ is one of $\{0, -\}$ and $?$ is one of $\{0, +, -\}$ (hence $A$ is PEP).

iii. $A \geq 0$ and $A$ is irreducible.
Thank You!