Riemann Problems for Two Dimensional Euler Systems

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Euler Equation

\[
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u + pI) = 0, \\
(\rho E)_t + \nabla \cdot (\rho E u + p u) = 0.
\end{cases}
\]  

(1)

Cauchy problems: Open
We look for self-similar solutions

$$(u, v, p, \rho)(\xi, \eta)$$

for

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}.$$
The Euler system becomes the self-similar form

\[-\xi \rho \xi - \eta \rho \eta + (\rho u) \xi + (\rho v) \eta = 0\]
\[-\xi (\rho u) \xi - \eta (\rho u) \eta + (\rho u^2 + p) \xi + (\rho uv) \eta = 0\]
\[-\xi (\rho v) \xi - \eta (\rho v) \eta + (\rho uv) \xi + (\rho v^2 + p) \eta = 0\]
\[-\xi (\rho E) \xi - \eta (\rho E) \eta + (\rho uE + pu) \xi + (\rho vE + pv) \eta = 0.\] (2)
Any initial data has to be

\[(u, v, p, \rho)(0, x, y) = \text{Independent of radius}\]

The so-called Riemann problem.

It is a boundary-value problem: Initial data becomes boundary data at infinity.
Figure 1: Four–constant Riemann problem

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Figure 2. Proposed structure for the interaction of four forward rarefaction waves in the 1990 paper. The characteristics are tangent to the sonic curve.
Numerical simulations:

1. Tong Zhang, Guiqiang Chen, Shuli Yang;
2. Lax Liu;
3. Schulz-Rinne, Collins, and Glaz;
4. Tadmore, Kurganov;
5. Glimm, J., Xiaomei Ji; Jiequan Li, Xiaolin Li; Peng Zhang; Tong Zhang; Yuxi Zheng;
6. W.C. Sheng et al
Figure 3: Density contour lines, courtesy of Lax and Liu.
4R (Two forward, two backward)
First, two wave interaction:

Hodograph method: paper with Jiequan Li, ARMA (vol 193(2009)).

Direct method (with Xiao Chen) (IUMJ, 59(2010)) with Jiequan Li and Zhicheng Yang, (JDE (2010)).

Then: Simple wave (With Tong Zhang and Jiequan Li: CMP, v.267(2006))
Then: Semi-hyperbolic patches
Semi-hyperbolic patch in channel flow
1. With Kyungwoo Song, on pressure gradient system, DCDS (2009)

2. With Mingjie Li (Ph.D student from CNU) ARMA 2011
Recently, in Tianyou Zhang’s thesis, we establish regularity via a new existence approach for the pressure gradient system

\[
\begin{align*}
\begin{cases}
u_t + p_x &= 0, \\
v_t + p_y &= 0, \\
pt + pu_x + pv_y &= 0,
\end{cases}
\end{align*}
\]

Or

\[
\left( \frac{pt}{p} \right)_t - pxx - pyy = 0.
\]
In the self-similar plane
\[(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0.\]

In polar coordinates,
\[(p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0.\]

\textit{Sonic line.}: \(r^2 - p = 0;\)

Hyperbolic where \(r^2 - p > 0;\) Elliptic in the region \(r^2 - p < 0.\)
D’Alembert type decomposition:
\[
\begin{align*}
\partial_+ \partial_- p &= q(\partial_+ p - \partial_- p) \partial_- p, \\
\partial_- \partial_+ p &= q(\partial_- p - \partial_+ p) \partial_+ p,
\end{align*}
\]
where
\[
\partial_\pm = \partial_\theta \pm \lambda^{-1} \partial_r,
\]
\[
\lambda = \sqrt{\frac{p}{r^2(r^2 - p)}},
\]
\[
q = \frac{r^2}{4p(r^2 - p)}.
\]
Introducing $R = \partial_+ p, S = \partial_- p, W = (p, R, S)$,

$$\begin{align*}
p_\theta &= \frac{R + S}{2}, \\
R_\theta - \lambda^{-1} R_r &= q(S - R)R, \\
S_\theta + \lambda^{-1} S_r &= q(R - S)S.
\end{align*}$$
For the semi-hyperbolic patch solution constructed in Song and Zheng, \( p_r \) may blow up. We do not expect better regularity. In numerical simulations, demarcation line between hyperbolic region and elliptic region is composed of the sonic line and a shock curve. Although the sonic curve changes dramatically close to the shock, for a large portion away from connection point with shock it is quite smooth and pressure \( p \) changes smoothly across it. We mainly focus on this part to propose our problem for today’s talk.
Problem 1: Given a sonic curve $r = f(\theta), \theta_a < \theta < \theta_b$, with

$$f' \geq \gamma > 0,$$

denoted by $\Gamma$. We look for solutions to the PGE in a class of functions with given first-order derivative $p_{\theta}(f^{-1}(r), r) = a_0(r)^2$ on $\Gamma$. We assume

$$a_0^2 \geq \alpha > 0.$$

As a result $p_r = 2f - \frac{p_\theta}{f} \equiv a_1(r)$ is also known on the sonic curve. We assume $p_{\theta\theta}$ is bounded at the sonic line.
We show that the precise boundary conditions are

\[
\begin{align*}
W|_\Gamma &= (r^2, a_0^2, a_0^2), \\
(p_\theta, \sqrt{r^2 - pR_\theta}, \sqrt{r^2 - pS_\theta})|_\Gamma &= (a_0^2, -\frac{a_0^2a_1}{2}, \frac{a_0^2a_1}{2}).
\end{align*}
\]

Using the hyperbolicity quantity \( t = \sqrt{r^2 - p} \) and \( r \) as new independent variables, and introducing new dependent variables

\[
\begin{align*}
U(t, r) &= R(t, r) - a_0^2(r) - a_1(r)t, \\
V(t, r) &= S(t, r) - a_0^2(r) + a_1(r)t,
\end{align*}
\]
The system is transformed into

\[ \begin{align*}
Ut + \frac{2t\lambda^{-1}}{S+2r\lambda^{-1}}Ur &= b_1(U, V), \\
Vt - \frac{2t\lambda^{-1}}{R-2r\lambda^{-1}}Vr &= b_2(U, V),
\end{align*} \]

where \( b_1(U, V) \) and \( b_2(U, V) \) contain a singular term \( \frac{U-V}{t} \) and depend on \( a_0 \) and \( a_1 \).

\[ \begin{align*}
b_1(U, V) &= \frac{1}{2} \left( \frac{U-V}{t} \right) + \left( \frac{2t^2qR}{S+2r\lambda^{-1}} - \frac{1}{2} \right) \left( \frac{U-V}{t} + 2a_1 \right) - f(a_0, a_1, V), \\
b_2(U, V) &= \frac{1}{2} \left( \frac{V-U}{t} \right) + \left( \frac{2t^2qS}{R-2r\lambda^{-1}} - \frac{1}{2} \right) \left( \frac{V-U}{t} - 2a_1 \right) + g(a_0, a_1, U),
\end{align*} \]
and

\[ f(a_0, a_1, V) = \frac{2t\lambda^{-1}}{S + 2r\lambda^{-1}} \left( \partial_r(a_0^2 + ta_1) \right), \]

\[ g(a_0, a_1, U) = \frac{2t\lambda^{-1}}{R - 2r\lambda^{-1}} \left( \partial_r(a_0^2 - ta_1) \right). \]

The boundary conditions are

\[ U(0, r) = V(0, r) = U_t(0, r) = V_t(0, r) = 0, \]

\((r_a \leq r \leq r_b).\)

By establishing an iteration scheme we establish the existence of a solution by showing that iteration sequence converges under a weighted metric
through characteristic analysis. The main result is stated as follows:

**Theorem.** Assume $a_0$ and $a_1$ are in $C^3([r_a, r_b])$ and $D(\delta_0)$ is a strong determinate domain, then there exists a $\delta > 0$ such that on $D(\delta)$, the degenerate hyperbolic problem has a classical solution in $S$.

**Proof.**

Now $\lambda^{-1}(t, r) = \frac{tr}{\sqrt{r^2 - t^2}}$ and $q(t, r) = \frac{r^2}{4(r^2 - t^2)t^2}$. 
Let
\[ D(\delta_0) := \{(t, r)| 0 \leq t \leq \delta_0, \quad r_1(t) \leq r \leq r_2(t)\}, \]
where \( r_1(t), r_2(t) \) are continuously differentiable on \( 0 \leq t \leq \delta_0 \), \( r_1(0) = r_a, r_2(0) = r_b \) and \( r_1 < r_2 \) for \( 0 \leq t \leq \delta_0 \).

**Definition** a strong determinate domain \( D(\delta_0) \).
Solutions of the Cauchy problem will be obtained within the class $S$ consisting of all continuously differentiable functions $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : D(\delta) \to \mathbb{R}^2$ satisfying the following properties:

(S1) $f_i(0, r) = \partial_t f_i(0, r) = 0 \quad (i = 1, 2)$,

(S2) $\| \frac{f_1}{t^2} \|_0 + \| \frac{f_2}{t^2} \|_0 \leq M$,

(S3) $\| \frac{\partial_r f_1}{t^2} \|_0 + \| \frac{\partial_r f_2}{t^2} \|_0 \leq M$, 
(S4) $F_r$ is Lipschitz continuous with respect to $r$ with
\[ \left\| \frac{\partial^2 f_1}{r^2} \right\|_0 + \left\| \frac{\partial^2 f_2}{r^2} \right\|_0 \leq M, \]

where $\| \cdot \|_0$ is the superem norm over domain $D(\delta)$ and $0 < \delta \leq \delta_0$. We will denote $W$ the larger class containing only continuous functions on $D(\delta)$ which satisfy the first two conditions (S1) and (S2). Both $S$ and $W$ are subsets of $C^0(D(\delta); \mathbb{R}^2)$. 
We define a new weighted metric on $\mathcal{S}$ and $\mathcal{W}$:

$$d(F, G) := \left\| \frac{f_1 - g_1}{t^2} \right\|_0 + \left\| \frac{f_2 - g_2}{t^2} \right\|_0.$$  \hspace{1cm} (3)

We will construct an integration iteration.

Assume $u(t, r)$ and $v(t, r)$ are admissible functions on $D(\delta)$, i.e. they belong to set $\mathcal{S}$. Differentiate along the two characteristics defined by:

$$\frac{d}{d_+(v)} := \partial_t + \Lambda_+(v)\partial_r, \quad \frac{d}{d_-(u)} := \partial_t + \Lambda_-(u)\partial_r.$$ \hspace{1cm} (4)
The iteration:

\[ \frac{d}{d_+(v)} U = b_1(u, v), \quad \frac{d}{d_-(u)} V = b_2(u, v). \] (5)

A mapping is thus determined:

\[ T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}. \]

Important properties of mapping \( T \) are given in the following lemmas.
Lemma 1. There exist positive constants $\delta$, $M$ and $0 < \beta < 1$ such that: $T$ maps $\mathcal{S}$ into $\mathcal{S}$ and for any pair $\mathbf{F}, \hat{\mathbf{F}}$ in $\mathcal{S}$, there holds $d\left( T(\mathbf{F}), T(\hat{\mathbf{F}}) \right) \leq \beta d(\mathbf{F}, \hat{\mathbf{F}})$. The constants depend only on the $C^3$ norms of $a_0$, $a_1$, $\alpha$ and $D(\delta_0)$.

For any $\mathbf{F}^{(1)} \in \mathcal{S}$, let $\mathbf{F}^{(n)} = T\mathbf{F}^{(n-1)}$, then the iteration sequence $\{\mathbf{F}^{(n)}\}$ is Cauchy in $(\mathcal{W}, d)$ which is a complete metric space.

The completeness of $(\mathcal{W}, d)$ can be proved by showing $d$ is a metric and the Cauchy sequence
in \((W,d)\) is actually convergent to an element in the same space; all these are elementary arguments. However since \((S,d)\) is not complete, the limit is only guaranteed to be in \(W\) but might not stay in \(S\). The limit will become the solution if it further has certain regularity say differentiable, which comes out of equi-continuity of the iteration sequence \(\{F^{(n)}\}\) which is guaranteed by following lemmas.
Lemma 2. The iteration sequence \( \{F^{(n)}\} \) has property that \( \{\partial_r F^{(n)}\} \) and \( \{\partial_t F^{(n)}\} \) are uniformly Lipschitz continuous on \( D(\delta) \).
Proof is complete: A smooth solution at a sonic curve exists.
Similar transformations are done for the steady and self-similar Euler systems.
Self-similar Euler: Use
\[ t = \sqrt{U^2 + V^2 - c^2/c}, \phi \]
where
\[ \phi_\xi = U = u - \xi, \phi_\eta = V = v - \eta \]
is the pseudo-potential.

For steady Euler: Use
\[ t = \sqrt{q^2 - c^2}, \text{ and } \phi \]
where \( \phi_x = u, \phi_y = v \).
Steady Euler in \((t, \phi)\):

\[
\partial_t R - \frac{t^2 \mu^2}{c^2 S} \partial_\phi R = -\frac{Rq^2}{2Sc^2 t} \{ R + S + [2(m+1) \cos^4 \omega - (m+3) \cos^2 \omega] S \}
\]

\[
\partial_t S - \frac{t^2 \mu^2}{c^2 R} \partial_\phi S = -\frac{Sq^2}{2Rc^2 t} \{ S + R + [2(m+1) \cos^4 \omega - (m+3) \cos^2 \omega] R \}
\]

\[
\partial_t c - \frac{2t^2 \mu^2}{c^2 (R+S)} \partial_\phi c = -\frac{t\mu^2}{c}.
\]

Here \(m = (3 - \gamma)/(\gamma + 1)\), \(\omega = (\alpha - \beta)/2\), \(\tan \alpha = \Lambda_+\), \(\tan \beta = \Lambda_-\), \(\mu^2 = (\gamma - 1)/(\gamma + 1)\), \(\cos^2 \omega = t^2/q^2\), \(R = \bar{\partial}^+ c/c\), \(S = \bar{\partial}^- c/c\).
Thank you.