Justification of the NLS Approximation for a Quasi-linear water waves model

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Abstract

I will describe an approximation theorem of a model for water waves which proves that the nonlinear Schrödinger equation (NLS) describes the evolution of wave packets for long periods of time. A key step is to derive a normal form transformation for the equation and in this talk I will focus on the presence of resonances, and how they can be dealt with.

This is joint work with Guido Schneider of the University of Stuttgart

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Introduction

- Modulation equations focus on important features of the solutions of complicated PDE’s without attempting to exactly solve the equation.
- Instead, we look for an approximation that accurately describes the true solution over physically relevant time-scales.
- A variety of such modulation equations exist for water waves:
  - The free boundary makes the problem difficult to treat exactly.
  - Different physical regimes lead to different modulation equations.
The NLS approximation

- Focus on wave packets.

- We want to know how the envelope evolves.
- Zakharov (1968) derived (nonrigorously) the NLS approximation as a modulation equation for the envelope function.
- Difficult to justify rigorously because of the long-time scales involved.
Recent results

- I’ll describe an approach to this problem due to Guido Schneider and myself (JDE vol. 251, pp. 238-269 (2011)). It allows one to prove the validity of the NLS approximation to a model for the problem of water waves on a channel of finite depth. We are currently working to extend it to the full water wave problem with Wolf-Patrick Düll.

- Recently N. Totz and S. Wu (“A rigorous justification of the modulation approximation to the 2D full water wave problem”) have proven that the NLS approximation is valid for water wave problem on a domain of infinite depth.
General Approach to Justifying Modulation Equations

Consider a system of PDE’s in the abstract form:

$$\partial_t u = \Lambda u + N(u, u)$$

Suppose that we believe that $u \approx \varepsilon \Psi$. We can write

$$u = \varepsilon \Psi + \varepsilon^\beta R$$

- If $\beta > 1$ and $\|R\| \sim \Theta(1)$ over the appropriate time scale, then $\varepsilon \Psi$ does provide the leading order approximation to the solution.
Equation for the Remainder

\[ \partial_t R = \Lambda R + 2\epsilon N(\Psi, R) + \epsilon^\beta N(R, R) + Res \]

Here, \( Res \) is an inhomogeneous term that measures the amount by which \( \Psi \) fails to be an exact solution - it can be made small by an appropriate choice of \( \Psi \).

Problems:

- The linear term \( \epsilon N(\Psi, R) \): This can lead to growth of the form \( \exp(C\epsilon t) \), which over the long time scales needed in the NLS regime \( (t \sim \mathcal{O}(\epsilon^{-2})) \) can cause us to lose control of our approximation.

- The nature of the nonlinear term: In the water wave problem, the nonlinearity is quasi-linear which means the IVP for \( R \) is complicated.
Controlling the Remainder

- We make normal form transformations to the equation to remove the $O(\varepsilon)$ linear terms.
  - These normal form transformations depend critically on the details of the spectrum of $\Lambda$, i.e. on the dispersion relation.
  - The form of the dispersion relation is related to the occurrence of resonances in the normal form.
Consider a *scalar* model equation:

\[ \partial_t^2 u = -\omega^2 u - \rho^2 u \]

- Here \( \omega \) and \( \rho \) are pseudo-differential operators.
- Choose their symbols so that:

\[ \hat{\omega}^2(k) = \rho^2(k) = k \tanh(k) . \]

This captures the following aspects of the water wave problem:

- Quadratic, quasi-linear nonlinearity.
- Same resonance structure in the linear part.
The construction of the approximation $\Psi$ is standard and causes no difficulty - we focus on the equation for the remainder.

It’s simpler to work with the equation in the form of a first order system

\begin{align*}
\partial_t u &= -\omega v \\
\partial_t v &= \omega u + \omega u^2.
\end{align*}
Diagonalize the system

When we study normal forms it’s convenient to diagonalize the linear part of the equation:

$$\partial_t U = \tilde{\Lambda} U + N(U, U)$$

where $U = (u_1, u_2)^T$, and

$$\tilde{\Lambda}(k) = \begin{pmatrix} i\omega_1(k) & 0 \\ 0 & i\omega_2(k) \end{pmatrix},$$

where $\omega_1(k) = -\omega_2(k) = \sqrt{k \tanh(k)}$. The nonlinear terms are written as convolutions

$$(\tilde{N}(\hat{u}, \hat{v})(k))_j = \int \sum_{m,n \in \{1,2\}} \tilde{\alpha}_{mn}^j(k, k - \ell, \ell)\hat{u}_m(k - \ell)\hat{v}_n(\ell)d\ell,$$  \hspace{1cm} (1)

where the kernel functions are related to the dispersion relation $\omega(k)$. 

SIAG PDE’s
Justification of NLS
Normal forms for systems with continuous spectrum

- As with normal forms for finite dimensional systems, *resonances* are key.
- Their role may be different (and “less serious””) in systems with continuous spectrum. (See “How real is resonance” H. McKean, CPAM vol. 50, pp. 317-322.)

Consider the equation for the remainder term in our approximation:

\[ \partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^\beta N(R, R) + \text{Res} \]

For the time being, focus on the linear part of the equation:

\[ \partial_t R = \Lambda R + 2\varepsilon N(\Psi, R) \]
Normal form transformations

Try to eliminate the terms $O(\varepsilon)$ by a normal form transformation:

$$\tilde{R}_j = R_j + \varepsilon B_j(\Psi, R) j = 1, 2$$

We choose the form of $B_j$ to mimic the form of the terms we want to eliminate:

$$B_{j_1}(\Psi, R) = \sum_{j_2, j_3=1}^{2} \hat{b}_{j_1; j_2, j_3}(k, k - \ell, \ell) \hat{\psi}_{j_2}(k - \ell) \hat{R}_{j_3}(\ell)$$
Normal form transformations

Inserting this into our equation, we find that the kernel of the normal form transformation should satisfy:

$$
\hat{b}_{j_1;j_2;j_3}(k, k - \ell, \ell) = \frac{\hat{\alpha}_{j_1;j_2;j_3}(k, k - \ell, \ell)}{\omega_{j_1}(k) - \omega_{j_2}(k - \ell) - \omega_{j_3}(\ell)}
$$

Two sources of difficulty:

- Resonance - i.e. points where the denominator vanishes.
- Loss of smoothness.
Resonances

To analyze the resonances in this problem note that due to the fact that the support of $\Psi$ is localized near $k_0$ in Fourier space means the principal contribution to the integral defining the normal form comes for points where $k - \ell \approx k_0$

Two types of resonance remain:

- Resonances at $k = 0$: If $k = 0$, the denominator is approximated by

$$\omega_{j_2}(-k_0) + \omega_{j_3}(k_0)$$

which vanishes if $j_2 = j_3$.

- We deal with this resonance by recalling that the kernel $\hat{\alpha}$ in the nonlinear term comes from the dispersion relation which means that the numerator of the normal form transform also vanishes at $k = 0$!
Resonances

The second type of resonance is:

- Resonances at $k = k_0$ In this case the denominator of normal form is approximately

$$\omega_{j_1}(k) - \omega_{j_2}(k_0) - \omega_{j_3}(k - k_0)$$

In this case if $j_1 = j_2$, the denominator vanishes when $k = k_0$.

- We deal with this resonance by noting that this resonance corresponds to having $\ell = 0$. This means that in the numerator of the normal form transform, we have $\hat{R}_{j_3}(\ell)$ with $\ell \approx= 0$. However, we can show that $\hat{R}_{j_3}(\ell) \approx 0$ when $\ell \approx= 0$. Thus, we don’t have to eliminate this term.
Loss of smoothness

A second problem is that even if the normal form transform exists, it may lose smoothness:

\[ \tilde{R}_j = R_j + \varepsilon B_j(\Psi, R) \]

but if \( R \in H^2 \), we can only show that \( \|B_j(\Psi, R)\|_{H^{s'}} \) is bounded if \( s < s' \).

This causes two different problems:

- It is difficult to invert the normal form transformation.
- It complicates the IVP for the error.
Loss of smoothness

In our problem, we loose half a derivative. Recall that the numerator of the kernel for the normal form transformation was:

\[ \hat{\alpha}_{j_1;j_2;j_3}(k, k - \ell, \ell) \sim \sqrt{|k|} \]

as \(|k| \to \infty\).

Thus, if \(R_j \in H^s\), we obtain

\[ \|B_j(\Psi, R)\|_{H^{s-1/2}} \leq C\|R\|_{H^s}. \]  \hspace{1cm} (2)
Inverting the transform

Inverting the transformation:
Standard approach:

\[ \tilde{R}_j = R_j + \varepsilon B_j(\Psi, R) \rightarrow R_j = \tilde{R}_j - \varepsilon B_j(\Psi, \tilde{R}) + \ldots \]

Instead we use energy estimates to invert the transformation:

Model:

\[ \tilde{R}_j = R_j + \varepsilon a(x) \partial_x R_j \]

(This would loose a full derivative - even worse than our example.)
Inverting the transform

Then

\[
\int \left( \partial_x^{s} \tilde{R}_j \right) \left( \partial_x^{s} R \right)_j = \int \left( \partial_x^{s} R \right)_j^2 + \varepsilon \int \partial_x^{s} R_j \partial_x^{s} (a(x) \partial_x R) \\
= \| \left( \partial_x^{s} R \right)_j \|_{L^2}^2 - \varepsilon \int a' (x) \left( \partial_x^{s} R \right)_j^2 + \text{l.o.t.}
\]

From this we conclude that

\[
\| \tilde{R} \|_{H^s} \geq (1 - \varepsilon C(a)) \| R \|_{H^s}
\]

so

- The transform is 1-1 and invertible
- We can bound the $H^s$ norm of $\tilde{R}$ in terms of the $H^s$ norm of $R$
The IVP

If we make the normal form transformations we obtain an evolution equation for $\tilde{R}$:

$$\partial_t \tilde{R} = \Lambda \tilde{R} + \varepsilon^\beta \tilde{N}(\tilde{R}, \tilde{R}) + \mathcal{O}(\varepsilon^2)$$

**Good News:**
- No more $\mathcal{O}(\varepsilon)$ terms.

**Bad News:**
- We loose “too much” smoothness.

$$\|\tilde{N}(\tilde{R}, \tilde{R})\|_{H^{s-1}} \leq C \|\tilde{R}\|_{H^s}^2$$
Introduce some “artificial smoothing”.

- No loss of generality to assume our NLS approximation is analytic.
- Look for solutions of the equation for the remainder in spaces of analytic functions.
- Introduce new dependent variables:
  \[
  \hat{R}_j(k, t) = e^{k(a - b\epsilon^2 t)} \hat{w}(k, t)
  \]
- Requires that \( R \) be analytic in a strip of width \( \Theta(1) \)
The new variable $w$ satisfies the equation:

$$\partial_t w = (\Lambda - b\varepsilon^2 |k|)w + \ldots$$

- The linear part of the equation now smooths!

If we consider $\|w\|_{\dot{H}^s} = \int |k|^{2s} |\hat{w}(k)|^2 dk$. Then

$$\frac{1}{2} \partial_t \|w\|_{\dot{H}^s}^2 = -b\varepsilon^2 \|w\|_{\dot{H}^{s+1/2}} + \varepsilon^\beta \|w\|_{\dot{H}^{s+1/2}}^2 \|w\|_{\dot{H}^{s-1/2}} + \ldots$$

These allow us to control the $H^2$ norm of $w$, which in turn controls $\tilde{R}$ in a space of analytic functions for a finite time – i.e. $t \sim O(\varepsilon^{-2})$, which is the time scale needed for the NLS approximation.
The main result

Where does all this leave us? Colloquially, one can state our main theorem as saying that if one takes any solution of the NLS there is a solution of our original equation which is well approximated by the NLS solution for times \( t \sim \mathcal{O}(\varepsilon^2) \). More precisely, we have:

\begin{equation*}
\textbf{Theorem}
\end{equation*}

\textit{For all } \kappa_0 \neq 0 \text{ and for all } C_1, T_0 > 0 \text{ there exist } T_1 > 0, C_2 > 0, \varepsilon_0 > 0 \text{ such that for all solutions } A \in C([0, T_0], H^6(\mathbb{R}, \mathbb{C})) \text{ of the NLS equation with }
\end{equation*}
\begin{equation*}
\sup_{T \in [0, T_0]} \|A(\cdot, T)\|_{H^6(\mathbb{R}, \mathbb{C})} \leq C_1
\end{equation*}

\text{the following holds. For all } \varepsilon \in (0, \varepsilon_0) \text{ there exists a solution of our water wave model which satisfies}

\begin{equation*}
\sup_{t \in [0, T_1/\varepsilon^2]} \left\| \begin{pmatrix} u \\ v \end{pmatrix} (\cdot, t) - \varepsilon \Psi(\cdot, t) \right\|_{(C^0_b(\mathbb{R}, \mathbb{R}))^2} \leq C_2 \varepsilon^{3/2},
\end{equation*}
Future Work

• We hope that this model captures most of the difficulties in the actual finite-depth water wave problem and are currently working with Wolf-Patrick Düll to extend these results to the true physical equations.
  • The system of equations is more complicated - in the form we write it it is a system of three PDE’s - this results in more resonances and many more terms in the normal form transformation.
  • The nonlinearity is more complicated, and in particular there are some cubic terms that must be eliminated - once again, this leads to more complicated resonance relations.
• The question of normal forms for systems whose linear part contains continuous spectrum is still very poorly understood in general, although there are a number of problems in which this method has been used. In particular, it is possible that the normal form is well defined even if there is a resonance in the integrand. Can one develop some more general theory for at least some class of systems.