Rate of convergence rate for vanishing viscosity approximations for hyperbolic balance laws

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OUTLINE:

1. Hyperbolic Balance Laws: Assumptions
2. Hyperbolic Conservation Laws i.e. $g(u) = 0$.
3. Major Challenges
4. Methods of Construction of Solution
5. Spreading of Rarefaction Waves
6. Main Results
7. Applications
Hyperbolic Balance Laws

The model

\[
\partial_t u + \partial_x f(u) + g(u) = 0 \quad (1)
\]

\[u(t, x) \in \mathbb{R}^n, \ x \in \mathbb{R}\]

\[f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n \] smooth

**H.1** (1) is strictly hyperbolic, that is

\[A(u) = Df(u)\] has \(n\) real and distinct eigenvalues

\[
\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u) \quad (2)
\]

known as characteristic speeds

\[< l_i(u), r_i(u) > \] – the corresponding left and right linearly independent eigenvectors

\[
l_i(u)r_j(u) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (3)
\]
Def 1. (1) is genuinely nonlinear, if for each $i = 1, \ldots, n,$

$$D\lambda_i(u)r_i(u) \equiv 1.$$ 

Def 2. (1) is linearly degenerate, if for each $i = 1, \ldots, n,$

$$D\lambda_i(u)r_i(u) \equiv 0.$$
Hyperbolic Systems of Conservation Laws

\[
\begin{aligned}
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) &= 0 \quad x \in \mathbb{R} \\
u(0, x) &= u_0,
\end{aligned}
\]  

(4)

where \( u = u(t, x) \in \mathbb{R}^n \) is the conserved quantity and

\[ F : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ smooth flux.} \]

The system is strictly hyperbolic:

\[ A(u) = Df(u) \text{ has } n \text{ real distinct eigenvalues} \]

\[ \lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u), \]  

(5)

and \( n \) linearly independent right eigenvectors \( r_i(u), i = 1, \ldots, n \).
Major challenges

♦ **Classical solutions**: Smooth solutions break down in finite time!
♦ **Weak solutions**: Uniqueness is lost!!!

**Admissibility criteria**: Second Law of Thermodynamics

\[
\forall (\eta, q) \text{ entropy-entropy flux pair}
\]

\[
\eta(u)_t + q(u)_x \leq 0 \text{ in } D'
\]

or equivalently as the “vanishing viscosity” limit of a family:

\[
\begin{cases}
\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon (B(u^\varepsilon)u^\varepsilon)_x & x \in \mathbb{R} \\
u^\varepsilon(0, x) = u_0,
\end{cases}
\]

(6)
Methods of construction of weak solutions

Results for small data:
1. Weak solutions of bounded variation \( BV \):
   - Random Choice Method (Glimm)
   - Front Tracking Method (Dafermos, Bressan, Holden–Risebro)
   - Vanishing Viscosity Method (Bianchini–Bressan)

2. Weak solutions in \( L^p \) space.
   - Compensated Compactness (Tartar, DiPerna, G.-Q. Chen and Serre)

Results for large data: Hold for special equations.
For systems of balance laws $g \neq 0$

we expect blow up of solutions in finite time
The presence of the production term $g(u)$

\[ \Downarrow \]

small oscillations in the solution may amplify in time

\[ \Downarrow \]

we do not have in general long term stability in BV

**Reference:** Dafermos and Hsiao '82

Local in time existence in BV – Random Choice Method

**Remark:** Global Existence is expected if $g$ satisfies a dissipation mechanism.
Consider a constant equilibrium solution $u^*$ to (1), i.e. $g(u^*) = 0$.

$$v_{i,t} + \lambda_i(u^*)v_{i,x} + \sum_{j=1}^{n} B_{ij}(u^*)v_j = 0,$$

(7)

where $B_{ij}$ are the entries of the $n \times n$ matrix

$$B(u) = [r_1(u), ..., r_n(u)]^{-1}Dg(u)[r_1(u), ..., r_n(u)].$$

(8)

H3. The matrix $B(u^*)$ is strictly column diagonally dominant i.e. there is a positive constant $\beta$ such that for all $i = 1, ..., n$,

$$B_{ii}(u^*) - \sum_{j \neq i} |B_{ji}(u^*)| > \beta > 0.$$ 

(9)
Under this Dissipativeness hyp., strict hyperbolicity and

\[ TVu_0 < < 1, \quad \| u_0 - u^* \|_{L^1} < < 1, \]

\[ \exists \text{ an admissible weak solution } u(t, x) \in \mathbb{R}^n \text{ to} \]

\[
\begin{cases}
\partial_t u + \partial_x f(u) + g(u) = 0, & x \in \mathbb{R}, \ t > 0 \\
u(0, x) = u_0(x),
\end{cases}
\]  

(10)

with

\[
TVu(t, \cdot) \leq Ce^{-\beta t} TVu_0, \quad t > 0
\]

\[
\| u(t) - v(t) \|_{L^1} \leq Le^{-\beta t} \| u_0 - v_0 \|_{L^1}, \quad t > 0
\]

\[
\| u(t) - u(s) \|_{L^1} \leq L'e^{-\beta s} |t - s|, \quad s < t
\]
Under this Dissipativeness hyp., strict hyperbolicity and 
\[ TVu_0 \ll 1, \quad \|u_0 - u^*\|_{L^1} \ll 1, \]
\[ \exists \text{ an admissible weak solution } u(t, x) \in \mathbb{R}^n \text{ to} \]
\[
\begin{cases}
\partial_t u + \partial_x f(u) + g(u) = 0, & x \in \mathbb{R}, \ t > 0 \\
u(0, x) = u_0(x),
\end{cases}
\] (10)

with
\[ TVu(t, \cdot) \leq Ce^{-\beta t} TVu_0, \quad t > 0 \]
\[ \|u(t) - v(t)\|_{L^1} \leq Le^{-\beta t} \|u_0 - v_0\|_{L^1}, \quad t > 0 \]
\[ \|u(t) - u(s)\|_{L^1} \leq L'e^{-\beta s}|t - s|, \quad s < t \]

Also, treated:
- Non conservative systems
- Explicit dependence of \( f \) and \( g \) on \((x, t)\).
Under this Dissipativeness hyp., strict hyperbolicity and 
$TVu_0 \ll 1$, $\|u_0 - u^*\|_{L^1} \ll 1$,
$\exists$ an admissible weak solution $u(t, x) \in \mathbb{R}^n$ to

$$\begin{cases}
    \partial_t u + \partial_x f(u) + g(u) = 0, & x \in \mathbb{R}, \ t > 0 \\
    u(0, x) = u_0(x),
\end{cases} \tag{10}$$

with

$$TVu(t, \cdot) \leq Ce^{-\beta t} TVu_0, \quad t > 0$$

$$\|u(t) - v(t)\|_{L^1} \leq Le^{-\beta t} \|u_0 - v_0\|_{L^1}, \quad t > 0$$

$$\|u(t) - u(s)\|_{L^1} \leq L'e^{-\beta s}|t - s|, \quad s < t$$

Also, treated:

- Non conservative systems
- Explicit dependence of $f$ and $g$ on $(x, t)$.

A different hypothesis on $g$: nonzero char. speeds and $g(u, x) \to 0$ as $x \to \pm \infty$ T.P. Liu ’75....
Hyperbolic Balance Laws
Dissipation mechanisms
Existing Results

References

♦ Dafermos and Hsiao ’82 ⇒ Global Existence
Random Choice Method

♦ Amadori and Guerra ’02 ⇒ Global Existence and Stability
Front Tracking Method

♦ Christoforou ’06 ⇒ Global Existence and Stability
Vanishing Viscosity Method - general flux, nonconservative form

♦ Dafermos ’06 ⇒ Global BV Solutions for hyperbolic balance
laws with source term that is weakly dissipative.
Random Choice Method
\[ \partial_t u(x, t) + \partial_x f(u(x, t)) + g(u(x, t)) = 0 \]

rich family of entropies
The dissipative properties of \( g \) in the vicinity of the equilibrium state \( u^* \) are encoded in the matrix

\[ A = r^{-1}(u^*)g_u(u^*)r(u^*) \]

\[ r(u) = [r_1(u), \ldots, r_n(u)], \quad g_u(u) \text{ Jacobian matrix of } g(u). \]

- The equilibrium state \( u^* \) is \( L^1 \)-stable.
- The principal diagonal entries of \( A \) are all positive:

\[ A_{ii} > 0, \quad i = 1, \ldots, n. \]
The Operator Splitting Method

The algorithm is initiated, at $s = 0$, by approximating the initial data by a piecewise constant data. To pass from $t = t_s$ to $t = t_{s+1}$, we first solve approximately the ordinary differential equation

$$\partial_t u = g(u) \text{ on } (t_s, t_{s+1})$$

and then solve independently, the conservation law

$$\partial_t u + \partial_x f(u) = 0.$$
The spreading of rarefaction waves

- scalar case: Oleinik '63
- genuinely nonlinear systems: Glimm-Lax '71
- piecewise genuinely nonlinear systems: Liu '81
- $2 \times 2$ genuinely nonlinear systems: Dafermos '00, Trivisa '96
- genuinely nonlinear systems of $n$ conservation laws: Bressan and Colombo '98, Bressan and Yang '04
- scalar, one inflection point: Jenssen and Sinestrari '01
- $2 \times 2$ systems, one inflection point: Ancona and Marson
- $n$ systems of hcl with general flux: LeFloch and Trivisa '04
Scalar case: Oleinik condition

\[ f'(u(y, t)) - f'(u(x, t)) \leq \frac{y - x}{t} \quad x \leq y, \ t > 0 \]

" Bounds the growth of the characteristic speed"

\[ \Downarrow \]

" The characteristics can NOT spread out arbitrarily fast"

This estimate is related to questions of:

- **Uniqueness** – Oleinik ’61
- **Local error estimates for approximate schemes** – Tadmor ’91
- **Regularity** – Cheng ’86, DeLellis and Reviere ’03
Genuinely nonlinear systems $g(u) = 0$

: 

$$\mu^i_{+T} \leq C \frac{b-a}{T} + C[Q(u(0)) - Q(u(T))]$$

This estimate was used to establish:

- Large time decay of solutions – Glimm-Lax ’71
- Up to a countable set of times $\{t_n\}_{n \in \mathbb{N}}$ the function $u(t)$ is a function of *special bounded variation* (SBV), that is, its distributional derivative $u_x$ is a measure with no cantorian part.

- Bianchini-Caravenna (2011) genuinely nonlinear systems
Hyperbolic Balance Laws

The spreading of rarefaction waves

- Non-genuinely nonlinear systems: LeFloch and Trivisa ’04

\[
\rho^i_T(I) \leq C \left( \frac{\text{meas}(I)}{t-s} + (Q(s) - Q(t)) + |V(s) - V(t)| \right)
\]  \hspace{1cm} (11)

for every interval \( I \subset \mathbb{R} \).

The \( i \)-rarefaction waves within an interval \([s, t]\) are of three types:

1. Waves already present at time \( s \) and which have propagated up to time \( t \):

\[
O(1) \frac{\text{meas}(I)}{t-s}.
\]

2. Waves generated by interactions which took place during the time interval \([s, t]\):

\[
O(1) \left( Q(s) - Q(t) \right).
\]

3. Waves cancelled during the time interval \([s, t]\), which are bounded by the change in total variation:

\[
O(1) |V(s) - V(t)|.
\]
Hyperbolic Balance Laws

The spreading of rarefaction waves

- systems of balance laws: Goatin and Gosse '01

\[ \partial_t u + \partial_x f(u) = g(x, u) \]

1. \( g : \mathbb{R} \times \Omega \to \mathbb{R}^n \) measurable w.r.t. \( x \), for \( u \in \Omega \), \( C^2 \) w.r.t. \( u \).
2. \( \| g(x, \cdot) \|_{C^2} \) bounded over \( \Omega \), uniformly in \( x \),
3. \( |g(x, u)| \leq \omega(x), \quad \| \nabla u g(x, u) \| \leq \omega(x), \ (x, u) \in \mathbb{R} \times \Omega. \)
   for \( \omega \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).
4. non-resonance condition.

\[ \mu_i^+(J) \leq C \cdot \frac{\text{meas}(J)}{t - s} + C[Q(u(s)) - Q(u(t))] + V(u_0) \cdot \| \omega \|_{L^1} \]
Our goal: Establish the rate of convergence for vanishing viscosity approximations for systems of hyperbolic balance laws.
A function of bounded variation $u(t, \cdot)$ implies $D_x u$ is a Radon measure. Define

$$
\begin{align*}
\mu^i &= l_i(u) \cdot D_x u, \\
\mu^i(\{x\}) &= \sigma_i,
\end{align*}
$$

on the sets of continuity of $u$ and at the points of jump of $u$.

Here, $\sigma_i$ denotes the strength of the $i$-wave in the Riemann solution to data $< u_l, u_r >$ with $u_l$ the left and $u_r$ the right states. If $u_l = u_0$, $u_1$, \ldots, $u_n = u_r$ are all the states in the Riemann solution, then we take the strength of the $i$-wave to be

$$
\sigma_i = \lambda_i(u_i) - \lambda_i(u_{i-1})
$$
Let $\mu, \mu'$ be two positive Radon measures. We say that

$$\mu \preceq \mu' \iff \sup_{\text{meas}(A) \leq s} \mu(A) \leq \sup_{\text{meas}(B) \leq s} \mu'(B), \text{ for every } s > 0.$$ (12)

**Remark:** "$\mu \preceq \mu'$" implies that $\mu'$ is more singular than $\mu$. Namely, it has a greater total mass, concentrated on regions with higher density.
We remark that the usual relation

$$\mu \leq \mu' \text{ if and only if } \mu(A) \leq \mu'(A) \text{ for every } A \subset \mathbb{R} \quad (13)$$

is stronger, in the sense that it implies $\mu \preceq \mu'$, but the reverse does not hold.
Glimm Functionals

\[ V(u) = \sum_{i} |\mu^i|(\mathbb{R}), \]

\[ Q(u) = \sum_{i<j} (|\mu^j| \otimes |\mu^i|) \{(x, y); x < y\} + \sum_{i} (\mu^i - \otimes |\mu^i|) \{(x, y); x \neq y\}. \]

Let \( u = u(x, t) \) be an entropy solution of the homogeneous system for (1). If the total variation of \( u \) is small and the constant \( C_0 \) is large enough, the functionals

\[ Q(t) = Q(u(t)), \quad \Upsilon(t) = V(u(t)) + C_0 Q(u(t)) \quad (14) \]

are nonincreasing in time.

The decrease in \( Q \) controls the amount of interaction, while the decrease in \( \Upsilon \) controls both the interaction and the cancellation in the solution.
Consider the system of balance laws

\[
\begin{cases}
\partial_t u + \partial_x f(u) + g(u) = 0 \\
u(x, 0) = u_0(x)
\end{cases}
\]  

(15)

- (15) is strictly hyperbolic
- (15) is genuinely nonlinear
- the dissipation hypothesis holds
Main Results

Theorem [C. Christoforou, K. Trivisa - JDE (2009)]

Let \( w = w(t, x) \) be the solution of the Cauchy problem for the scalar Burgers's equation with impulsive source term

\[
\partial_t w + \partial_x \left( \frac{w^2}{2} \right) = -\beta w - k \text{sgn}(x) \cdot \frac{d}{dt} Q(u(t)),
\]

\( w(0, x) = \text{sgn}(x) \cdot \sup_{\text{meas}(A) < 2|x|} \frac{\mu_0^+}{2} \)

with \( k \) denoting some positive constant.

Then, for every solution \( u = u(t, x) \) of (15) with small total variation, and for every \( t \geq 0 \), and \( i = 1, \ldots, n \),

\[
\mu_t^{i+} \leq D_x w(t).
\]
Consider the hyperbolic balance law

$$\partial_t u + \partial_x f(u) + g(u) = 0$$  \hspace{1cm} (19)

together with the viscous approximations

$$\partial_t u^\varepsilon + A(u^\varepsilon)u_x^\varepsilon + g(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon.$$  \hspace{1cm} (20)

Here $A(u) = Df(u)$ is the Jacobian matrix of $f$. Given I.D. $u(0,x) = \bar{u}(x)$ having small total variation the solutions \{u^\varepsilon\} of the (20) exist for all $t \geq 0$ have uniformly small total variation and converge to a unique solution of the (19) as $\varepsilon \to 0$. 

Aim: Estimate the distance $\|u^\varepsilon(t) - u(t)\|_{L^1}$.

Related work:

- Kuznetsov (1976): 1-d scalar conservation law
  \[ \|u^\varepsilon(\tau) - u(\tau, \cdot)\|_{L^1} = O(1) \cdot \varepsilon^{1/2}. \]


\[ \downarrow \]

Convergence rate $O(1)\varepsilon^\gamma$ for any $\gamma < 1$.

- Bressan and Yang (2003): For general BV solutions possibly with everywhere dense set of shocks

  \[ \|u^\varepsilon(\tau, \cdot) - u(\tau, \cdot)\|_{L^1} = O(1)(1 + \tau)\sqrt{\varepsilon} \ln \varepsilon \|TV\{\bar{u}\}. \]
**Theorem [C. Christoforou, K. Trivisa (2010)]**

Let the system (19) be strictly hyperbolic and assume that each characteristic field is genuinely nonlinear. Then, given any initial data \( u(0, \cdot) = \bar{u} \) with small total variation, for every \( \tau > 0 \) the corresponding solutions \( u \) and \( u^\varepsilon \) of (19) and (20) satisfy the estimate

\[
\| u^\varepsilon(\tau, \cdot) - u(\tau, \cdot) \|_{L^1} = 
\]

\[
O(1) \left( 1 + \frac{1 - e^{-\beta \tau}}{\beta} + e^{-\beta \tau} \right) \cdot \sqrt{\varepsilon} |\ln \varepsilon| TV\{\bar{u}\}
\]

\[
+ O(1) \varepsilon \| Dg \|_{L^\infty} (\| \bar{u} - u^* \|_{L^1} + \sqrt{\varepsilon} TV\{\bar{u}\}).
\]
Call $u$ the $\varepsilon'$-approximate front-tracking approximation and assume $\bar{u} = u(0)$ is piecewise constant.

**Step 1.** Starting from $u$ we construct an approximation $v = v(x, t)$ of the viscous solution $u^\varepsilon$ by taking a mollification $u \ast \varphi \sqrt{\varepsilon}$ and inserting viscous shock profiles at the locations of finitely many large shocks for each fixed $\varepsilon$.

$$v = u \ast \varphi \sqrt{\varepsilon} + \sum_{\alpha \in BS} (\tilde{\omega}_\alpha - \varrho_\alpha).$$

1. $v$ is smooth on each strip $[t^k_{i-1}, t^k_i) \times \mathbb{R}$.
2. $\|v(0) - \bar{u}\|_{L^1} = O(1) \cdot \delta_0 \sqrt{\varepsilon}$,
3. $\|v(\tau) - u(\tau)\|_{L^1} = O(1) \cdot \delta_0 \sqrt{\varepsilon} e^{-\beta \tau}$,
Step 2. We obtain error estimates by introducing new Liapunov functionals that control interactions of shock waves in the same family and also interactions of waves in different families.
Error estimate

Let $S$ be a Lipschitz continuous semigroup:

$$S : \mathcal{D} \times [0, \infty) \mapsto \mathcal{D},$$

$$\|S_t w(0) - w(t)\|_{L^1} \leq L \int_0^t \liminf_{h \to 0^+} \frac{\|S_h w(\tau) - w(\tau + h)\|_{L^1}}{h} \, d\tau,$$

(21)

where $L$ is the Lipschitz constant of the semigroup and $w(\tau) \in \mathcal{D}$. The above inequality appears extensively in the theory of front tracking method: e.g.

(i) the entropy weak solution by front tracking coincides with the trajectory of the semigroup $S$ if the semigroup exists,

(ii) uniqueness within the class of viscosity solutions, etc.

References: Bressan et al.
We show that the approximation $v = v(t,x)$

$$v = u \ast \varphi_\delta + \sum_{\alpha \in BS} (\tilde{\omega}_\alpha - \rho_\alpha).$$

has the following properties:

Let $t_1^k < t_2^k < \cdots < t_{N_k}^k = \tau$ be the interaction times in the front tracking solution $u$. Then $v$ is smooth on each strip $[t_{i-1}, t_i) \times \mathbb{R}$, but discontinuous at the time discretization $t = t_k^*$ of the operator splitting. Let $\delta_0 = TV(\tilde{u})$,

$$\int_{k_s}^{(k+1)s} \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}| dx dt =$$

$$O(1) \left(1 + \frac{1 - e^{-\beta \tau}}{\beta}\right) \cdot \delta_0 \sqrt{\varepsilon} |\ln \varepsilon|,$$
\[ \sum_{k} \sum_{1 \leq i \leq N_k} \int |v(t_i^k, x) - v(t_i^k-, x)| \, dx = O(1) \cdot \delta_0 \sqrt{\varepsilon} \ln \varepsilon. \]

\[ \sum_{k} \int |v(t_k^*, x) - v(t_k^*-, x)| \, dx = O(1)\varepsilon \|Dg\|_{\infty} \left( \|\bar{u} - u^*\|_{L^1} + \sqrt{\varepsilon} \delta_0 \right). \]
P-system with relaxation

\[ \partial_t u - \partial_x v = 0 \]
\[ \partial_t v + \partial_x p(u) + v - f = 0, \]

where \( p \) and \( f \) satisfy the subcharacteristic condition

\[ -p_u(u) > f_u^2(u) \]

Dissipativiness Hypothesis Fails!
In many physical systems \( \Rightarrow \) Weak Dissipation
Dafermos (2006): For such systems a special change of variables \( \Rightarrow \) a new system for which the Dissipativiness Hypothesis holds true.
Future plans

- Investigate the relevance of the dissipative mechanism presented here with hyperbolic balance laws in nonlinear elasticity.
- Relaxation schemes for hyperbolic balance laws.
- Most of the results refer to small initial data. What can we say about hyperbolic balance laws with large I.D.?
- Can the condition of genuine nonlinearity be relaxed?
- Investigate various other conditions that appear in the literature such as the Kawashima condition.