Two Phase Flow in Porous Media: Stability of Fronts

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SIAM APDE 2011

Supported by NSF grants DMS 0604047, DMS 0968258, DMS 0636590 RTG
Secondary oil recovery: water flood

Buckley and Leverett, 1942
scalar 1st order PDE in 1-d

Gray and Hassanizadeh, 1990’s
dynamic capillary pressure: rate dependence

Plane waves: undercompressive shocks; Pop, Cuesta, Peletier et al 2006-2011

Fingering instability: Saffman, Taylor, 1958
New insights; Yortsos and Hickernell, 1989
2-D Model  \( w: \text{water}; \ o: \text{oil}; \ T: \text{total}. \)

\[ u(x, y, t): \text{saturation} \ (\text{vol. fraction}) \ of \ \text{water}, \ (1 - u): \text{oil saturation} \]

\[ p(x, y, t): \text{pressure} \ \text{in water} \]

Conservation of mass with Darcy’s law: velocity \( \mathbf{v} = -\lambda^w(u) \nabla p \):

\[
\phi \frac{\partial u}{\partial t} - \nabla \cdot (\lambda^w(u) \nabla p) = 0 \quad \phi = \text{porosity}, \ \lambda(u) = \frac{K k(u)}{\mu}
\]

\( K = \text{absolute permeability}, \ k(u) = \text{relative permeability}, \ \mu = \text{viscosity} \)

Incompressibility: \( \nabla \cdot \mathbf{v}^T = 0 \)

\[
\nabla \cdot \left( \lambda^T(u) \nabla p + \lambda^o(u) \nabla p_c(u) \right) = 0
\]

\( \lambda^T = \lambda^w + \lambda^o, \quad p_c(u) = p^o - p^w : \text{capillary pressure}; \)

\textit{For simplicity, neglect gravity}
Plane waves

\[ p_e = p_e(u) : \text{equilibrium capillary pressure; decreasing function} \]

One-dimensional equation: \( \partial_x \mathbf{v}^T = 0 : \mathbf{v}^T = (V, 0) \) constant

eliminate pressure gradient \( \partial_x p \)

Relative permeability functions: \( k^w(u) = \kappa^w u^2; \quad k^o(u) = \kappa^o (1 - u)^2 \)

\[ u_t + f(u)x = - (p'_e(u)u_x)_x \]

\[ f(u) = V \frac{u^2}{u^2 + M(1 - u)^2} \]

\[ M = \frac{m^o}{m^w} \text{ mobility ratio; } m^j = \frac{\kappa^j}{\mu^j} \]
Gray and Hassanizadeh (1990, 1993) propose that capillary pressure should be rate dependent:

\[ p_c(u, u_t) = p_e(u) - \tau u_t \]

\( p_e \): equilibrium capillary pressure; \( p_e(u) = -u \) for simplicity

DiCarlo: Water Resources Research, 2004: Experiments (with gravity) show nonmonotonic saturation profiles
Modified Buckley-Leverett Equation (1-D)

\[ u_t + f(u)_x = (H(u)u_x)_x + \tau (H(u)u_{tx})_x \]

\[
f(u) = V \frac{u^2}{u^2 + M(1-u)^2}
\]

\[
H(u) = \frac{u^2(1-u)^2}{u^2 + M(1-u)^2}
\]

\[ M = m^o/m^w \text{ mobility ratio; } m^j = \kappa^j/\mu^j \]

![Graph of f(u) and H(u) for M=2](image)
Scalar conservation law: $u_t + f(u)_x = 0$

**Idealization:** no capillary pressure; characteristic speed $f'(u)$

Scale invariant solutions: building blocks for solving initial value problem

**Rarefactions**

$$u(x, t) = \begin{cases} 
  u_- & \text{if } x < f'(u_-)t \\
  r\left(\frac{x}{t}\right) & \text{if } f'(u_-)t \leq x \leq f'(u_+)t \\
  u_+ & \text{if } x > f'(u_+)t 
\end{cases}$$

**Shocks**

$$u(x, t) = \begin{cases} 
  u_- & \text{if } x < st \\
  u_+ & \text{if } x > st 
\end{cases}$$

Rankine-Hugoniot condition: shock speed

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$
Admissible Shocks

\[ f'(u_+) < s < f'(u_-) \quad \text{[Lax]} \]

shock is admissible if there is a traveling wave from \( u_- \) (unstable node) to \( u_+ \) (saddle point).

Dynamic capillary pressure admits \underline{undercompressive} shocks \( \Sigma \): \( s > f'(u_{\pm}) \) PLUS corresponding traveling wave (saddle-to-saddle)

Buckley-Leverett solution 1942

Solve conservation law with initial jump from all water $u_- = 1$ to all oil $u_+ = 0$: water flooding:

$$u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Solution: rarefaction from $u_- = 1$ to $u^*$; Lax shock from $u^*$ to $u_+ = 0$: rarefaction-shock.
Solve conservation law with initial jump from $u_\ell$ to $u_r$

$$u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} 
  u_\ell & \text{if } x < 0 \\
  u_r & \text{if } x > 0
\end{cases}$$

R: Rarefaction Wave
S: Lax Shock
RS : Rarefaction - Shock
The Riemann Problem; dynamic capillary pressure

\[(RP) : \quad u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} u_ℓ & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}\]

R: Rarefaction Wave
S: Admissible Lax Shock
Σ : Undercompressive Shock
$u_\ell \in (u_{\text{mid}}, u_\Sigma)$:

- **Lax shock** $u_\ell$ to $u_\Sigma$
- **undercompressive shock** $u_\Sigma$ to $u_r$

$u_\ell \in (u_\Sigma, 1)$:

- **rarefaction wave** $u_\ell$ to $u_\Sigma$
- **undercompressive shock** $u_\Sigma$ to $u_r$
Classical Result: Saffman-Taylor Instability (1958)

Pure fluids: all water \((u = 1)\) displacing all oil \((u = 0)\)

No capillary pressure \((p_c \equiv 0)\) \(\Rightarrow\) sharp interface

Does not capture rarefaction-shock solution of Buckley-Leverett

\[
\begin{align*}
\text{WATER} & \quad \text{OIL} \\
\begin{array}{l}
u = 1 \\
p_- \\
v_- = V \\
u = 0 \\
p_+ \\
v_+ = V \\
x = Vt
\end{array}
\end{align*}
\]

Perturb pressure and interface, 
\[
x = Vt + ae^{i\alpha y + \sigma t}
\]

wave number \(\alpha\), growth rate \(\sigma(\alpha)\)

Saffman-Taylor result:
\[
\sigma = \sigma_1 \alpha + h.o.t., \quad \sigma_1 = V \frac{1 - M}{1 + M}
\]

Fingering instability:
Mobility ratio \(M = \frac{\mu^w}{k^w} / \frac{\mu^o}{k^o} < 1\)
Saffman-Taylor Analysis: 1-dimensional base state

Interface $\bar{x} = Vt$; velocity $V = -m_\pm \partial_x \bar{p}_\pm$; mobility $m_\pm = k_\pm / \mu_\pm$

Continuous pressure $\bar{p}_\pm = -\frac{V}{m_\pm} (x - Vt) = -\frac{V}{m_\pm} z$, $z = x - Vt$;

shock location: $\bar{z} = \bar{x} - Vt = 0$
Saffman-Taylor perturbation analysis

2-d equations: $v_\pm = -m_\pm \nabla p_\pm$; $\nabla \cdot v_\pm = 0$

Thus, the pressure is harmonic: $\Delta p_\pm = \partial_z^2 p_\pm + \partial_y^2 p_\pm = 0$ \hspace{1cm} (1)

$p_\pm(z, y, t) = -\frac{V}{m_\pm} z + q_\pm(z) e^{i\alpha y + \sigma t}$ \hspace{1cm} interface: $z = \hat{z}(y, t) = ae^{i\alpha y + \sigma t}$

From (1): $q''_\pm - \alpha^2 q_\pm = 0$, $q_\pm(\pm \infty) = 0$ \hspace{1cm} (resp.)

Hence, $q_\pm(z) = b_\pm e^{\mp \alpha z}$

Next: continuity of velocity and pressure at interface $z = \hat{z}$
Saffman-Taylor dispersion relation $\sigma(\alpha)$

\[ p_{\pm}(z, y, t) = -\frac{V}{m_{\pm}} z + b_{\pm} e^{\mp \alpha z} e^{i\alpha y+\sigma t} \quad \text{interface: } z = \hat{z}(y, t) = ae^{i\alpha y+\sigma t} \]

Continuity of velocity and pressure at interface $z = \hat{z}$, retaining linear terms in coefficients $a, b_{\pm}$

Horizontal velocity:

\[ \frac{\partial x}{\partial t} = a\sigma e^{i\alpha y+\sigma t} + V = -m_{\pm} \frac{\partial p_{\pm}}{\partial z} \big|_{\hat{z}} = V \pm m_{\pm} b_{\pm} \alpha e^{i\alpha y+\sigma t} \]

Thus,

\[ b_{\pm} = \pm \frac{\sigma}{\alpha} a/m_{\pm} \]

Similarly, $p_{\pm} = p_{-}$ at $z = \hat{z}$:

\[ -\frac{V}{m_{+}} a + b_{+} = -\frac{V}{m_{-}} a + b_{-} \]

3 linear equations for $a, b_{\pm}$, parameter $\frac{\sigma}{\alpha}$

Nonzero solution:

\[ \frac{\sigma}{\alpha} = V \frac{1 - M}{1 + M} \]

\[ M = \frac{m_{\pm}}{m_{-}} = \frac{\mu_{-}}{k_{-}} / \frac{\mu_{\pm}}{k_{+}} < 1 \]
Stability of Lax shocks

Variable saturation $u = u(x, y, t)$, pressure $p(x, y, t)$  \( p_c \equiv 0 \), linearized equations

$$\sigma = \sigma_1 \alpha + \ldots \quad \sigma_1 = \sqrt{\frac{\lambda^T(u-) - \lambda^T(u_+)}{\lambda^T(u-) + \lambda^T(u_+)}}$$

\( \lambda^T = \) total mobility; shock \( u = u_{\pm} \); \( V = \) shock speed

Yortsos and Hickernell, 1989; stability of smooth traveling wave matched asymptotics (with \( p_c(u) \))

Conclusion: Long-wave stability \( \iff \lambda^T(u-) < \lambda^T(u_+) \)
2-dimensional stability

2-d equations with $p_c \equiv 0$ variables $u, p$ saturation, pressure:

$$\frac{\partial u}{\partial t} - \nabla \cdot (\lambda^w(u) \nabla p) = 0$$

$$\nabla \cdot (\lambda^T(u) \nabla p) = 0$$

(1)

Interface $x = \hat{x}(y, t)$, normal in $t, x, y : (-\hat{x}_t, 1, -\hat{x}_y)$

Jump condition at shock: $([q] = q_+ - q_-)$

$$-\hat{x}_t[u] - [\lambda^w(u)p_x] + \hat{x}_y[\lambda^w(u)p_y] = 0$$

$$-[\lambda^T(u)p_x] + \hat{x}_y[\lambda^T(u)p_y] = 0$$

(2)

Base shock: $u = \bar{u}_\pm, p = \bar{q}_\pm(x - Vt), \hat{x} = Vt$, constants $\bar{u}_\pm, \bar{q}_\pm, V$

$$V = \frac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-}, \quad f(u) = v^T \lambda^w(u) \quad \bar{q}_\pm = -\frac{v^T}{\lambda^T(\bar{u}_\pm)}$$
2-d stability: perturb variables and linearize equations

\[ u = \bar{u}_\pm + u_\pm(z)e^{i\alpha y+\sigma t}, \quad p = \bar{q}_\pm z + q_\pm(z)e^{i\alpha y+\sigma t} \]
\[ \hat{z} = \hat{x} - Vt = ae^{i\alpha y+\sigma t}, \quad z = x - Vt \]

Linearized equations: \( (\; = \frac{d}{dz}) \)

\[
\sigma u - Vu' - \lambda^w(\bar{u}_\pm)(q'' - \alpha^2 q) + \frac{d\lambda^w}{du}(\bar{u}_\pm)\bar{q}_\pm u' = 0 \\
\lambda^T(\bar{u}_\pm)(q'' - \alpha^2 q) + \frac{d\lambda^T}{du}(\bar{u}_\pm)\bar{q}_\pm u' = 0
\] (3)

Relevant solutions for small \( \alpha \):

\[ u = 0, \quad q_\pm = b_\pm e^{\mp \alpha z}, \quad \pm(z - \hat{z}) > 0, \quad - \text{as for Saffman-Taylor!} \]

Now linearize the jump conditions and find solvability condition for \( b_\pm, a \)
2-dimensional stability continued

\[ \sigma a[\bar{u}] + [\lambda^w(\bar{u})q'] = 0 \quad (4) \]

\[ [\lambda^T(\bar{u})q'] = 0 \quad (5) \]

Equation (5):

\[ \lambda^T(\bar{u}_+)b_+ = -\lambda^T(\bar{u}_-)b_- \quad (6) \]

Then (4) implies

\[ b_- \lambda^T(\bar{u}_-)(f(\bar{u}_+) - f(\bar{u}_-))v^T = -\frac{\sigma}{\alpha} a(\bar{u}_+ - \bar{u}_-) \]

But \((f(\bar{u}_+) - f(\bar{u}_-))/(\bar{u}_+ - \bar{u}_-) = V\), the shock speed, so

\[ b_- \lambda^T(\bar{u}_-)V = -a\frac{\sigma}{\alpha} v^T \quad (7) \]
2-dimensional stability continued

Third equation comes from continuity of pressure

\[ p = \bar{q}_\pm (x - Vt) + q_\pm (x - Vt)e^{i\alpha y + \sigma t}, \quad q_\pm = b_\pm e^{\mp z} \quad (\pm (z - \hat{z}) > 0) \]

at \( z = x - Vt = \hat{z}(y, t) \). Consequently,

\[ \bar{q}_+ a + b_+ = \bar{q}_- a + b_- \tag{8} \]

Thus,

\[ (\bar{q}_+ - \bar{q}_-) a = -\frac{\sigma v^T}{\alpha V} a \left( \frac{1}{\lambda^T(\bar{u}_+)} + \frac{1}{\lambda^T(\bar{u}_-)} \right), \quad (\text{from } (6,7)) \]

Since \( \bar{q}_\pm = -\frac{v^T}{\lambda^T(\bar{u}_\pm)} \), we obtain

\[ \frac{\sigma}{\alpha} = V \frac{\lambda^T(\bar{u}_-) - \lambda^T(\bar{u}_+)}{\lambda^T(\bar{u}_-) + \lambda^T(\bar{u}_+)} \]
Interpretation of stability condition: quadratic relative permeabilities: \( k(u) = \kappa u^2 \)

Lax shocks for \( u_+ < u_- \leq u^*_\alpha \)

Stability boundary: \( u_+ = -u_- + \frac{2M}{M+1} \)

\[ M = 0.2 \quad \frac{2M}{M + 1} = \frac{1}{3} \]

Inflection point \( I \) of \( f(u) \) at \( u_I = 0.2591 \)

S: Stable Lax shocks

U: Unstable Lax shocks

Undercompressive shocks are all unstable
Crank-Nicolson time step, centered difference spatial discretization, first-order upwind scheme for advection term with periodic side boundary conditions, moving frame

$$\Delta t = O(10^{-3}), \Delta x = \Delta y = O(10^{-2})$$

Initial condition: randomly perturbed hyperbolic tangent

$$u_- = 0.2, \ u_+ = 0, \ M = 0.05$$
Numerical Simulations - Stable case

\[ M = 0.2 \]

\[ u_- = 0.25, \ u_+ = 0 \]

Oil-water mixture displaces oil

Lax shock

Initial perturbation decays
Numerical Simulations: Unstable case: Fingering Instability

- $M = 0.2$
- $u_- = 0.25$, $u_+ = 0.15$
- Lax shock
- Initial perturbation grows $\Rightarrow$ fingering instability
Conclusions

Undercompressive 1-d shocks with dynamic capillary pressure: non-monotone solutions

Analysis of stability/fingering instability in 2-d, connection to Saffman-Taylor instability

Surprising linear dependence of growth rate on wave number for long waves: distinguishes stable waves from unstable

Numerical simulations of full parabolic/elliptic system; Riaz and Tchelepi (2006) also conducted numerical experiments

Oil/water mixture displacing oil can be stable.