The Dynamics of Perturbations of Minimal Mass Solitons

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Outline

1. The Nonlinear Schrödinger Equation
   - Soliton Stability
   - The ODE System
   - Numerical Methods
   - Results
   - Conclusions

2. The Generalized Korteweg-deVries Equation
   - Set Up
   - Results
   - Conclusions

3. Current and Future Work
The result discussed here is joint work with J. Marzuola (University of North Carolina, Chapel Hill) and G. Simpson (University of Minnesota). This first part of this work is available in Journal of Nonlinear Science, 2010, Volume 20, Number 4, Pages 425-461. The second part is a preliminary report.
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3. Current and Future Work
The Problem

We consider the initial value problem for the nonlinear Schrödinger equation (NLS) in $\mathbb{R} \times \mathbb{R}^+$:

\[
\begin{cases}
  iu_t + \Delta u + g(|u|^2)u = 0, \\
  u(x,0) = u_0(x),
\end{cases}
\]

(1)

where the nonlinearity $g(s)$ is a saturated nonlinearity of the form

\[
g(s) = s^{\frac{q}{2}} \left( s^{\frac{p-q}{2}} \frac{s^{\frac{p-q}{2}}}{1 + s^{\frac{p-q}{2}}} \right).
\]

(2)

and $p > 6 > 4 > q > 0$.

For $|u|$ large, (1) behaves as though it were $L^2$-subcritical while for $|u|$ small, it behaves as though it were $L^2$-supercritical. This guarantees both existence of soliton solutions and global well-posedness in $H^1$. 
Solitons

A soliton solution of (1) is a function $u(t, x)$ of the form

$$ u(t, x) = e^{i\omega t} \phi_\omega(x), \quad (3) $$

where $\omega > 0$ and $\phi_\omega(x)$ is a positive, radially symmetric, exponentially decaying solution of the equation:

$$ \Delta \phi_\omega - \omega \phi_\omega + g(\phi_\omega^2)\phi_\omega = 0. \quad (4) $$

For our particular nonlinearity, for any $\omega > 0$ there is a unique solitary wave solution $\phi_\omega(x)$ to (4), see [1] and [6].
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Conservation Laws

For data $u_0 \in H^1 \cap L^2(|x|^2)$, there are several conserved quantities.

Conservation of Mass (or Charge):

$$Q(u) = \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^d} |u_0|^2 \, dx.$$ 

Conservation of Energy: (Here $G(t) = \int_0^t g(s) \, ds$.)

$$E(u) = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} G(|u|^2) \, dx = \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx - \int_{\mathbb{R}^d} G(|u_0|^2) \, dx,$$

Of great importance is that $Q_\omega := Q(\phi_\omega)$ and $E_\omega := E(\phi_\omega)$ are differentiable with respect to $\omega$. Differentiating Equation (4), $Q$ and $E$ all with respect to $\omega$, we have the relation

$$\partial_\omega E_\omega = -\omega \partial_\omega Q_\omega.$$
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$$\partial_\omega E_\omega = -\omega \partial_\omega Q_\omega.$$
Definition of Stability

We are interested in the stability of these explicit solutions under perturbations of the initial data.

The soliton is said to be orbitally stable if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that, for any initial data $u_0$ such that $\|u_0 - \phi_\omega\| < \delta$, for any $t < 0$, there is some $\theta \in \mathbb{R}$ such that $\|u(x, t) - e^{i\theta} \phi_\omega(x)\| < \epsilon$.

The convexity of $\delta(\omega) := E_\omega + \omega Q_\omega$ determines stability: when positive, stability under small perturbation is guaranteed; when negative, exponential instability occurs.[9], [10]

At a critical point of $Q_\omega$, soliton instability is more subtle, because it is due solely to nonlinear effects. Comech and Pelinovsky showed that a purely nonlinear instability occurs, by approximating perturbed soliton solutions with an ODE that is unstable for certain initial conditions.[3]
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A plot of soliton stability curves for several nonlinearities

Figure: Plots of the soliton curves ($\phi(\omega)$ with respect to $\omega$) for a subcritical nonlinearity, critical nonlinearity, supercritical nonlinearity, and a saturated nonlinearity.
Motivation

Although [3] demonstrated that the minimal mass soliton has a purely nonlinear instability, we conjecture that on a longer time scale the solution will ultimately relax to the stable branch of the soliton curve.

The purpose of this work is to numerically explore this conjecture as follows:

- Following [3], we break the perturbation into the discrete and continuous parts relative to the linearization of the Schrödinger operator around the minimal mass soliton.
- The discrete portion yields a four dimensional system of nonlinear ODEs. We further simplify the system by Taylor expanding the equations in the dependent variables and dropping cubic and higher terms.
- We use a sinc-spectral method to numerically evaluate the coefficients of this Taylor expansion, and then we use a standard stiff solver in Matlab to solve the resulting system of ODEs.
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- We use a sinc-spectral method to numerically evaluate the coefficients of this Taylor expansion, and then we use a standard stiff solver in Matlab to solve the resulting system of ODEs.
We are interested in writing down ODEs for a small perturbation of a soliton, with the soliton and phase parameters allowed to modulate. We are assuming spherical symmetry of our data, so no other modulation parameters are required.

We begin with the ansatz

\[ \bar{u}(t) = e^{(\int_0^t \omega(t') dt' + \theta(t))} J(\phi\omega(t) + \rho(t)). \]  

(5)
The Final Quadratic System of ODEs

With our assumptions, we conclude the following:

The quadratic approximation for the evolution of a perturbation of the minimal mass soliton, given in (5), ignoring coupling to the continuous spectrum, is

\[ \dot{\theta} = -c_{14} \dot{\theta} \rho_4 + n_{133} \rho_3^2 + n_{144} \rho_4^2 + \dot{\omega} p_{13} \rho_3, \]
\[ \dot{\omega} = \rho_3 + c_{23} \dot{\theta} \rho_3 + n_{234} \rho_3 \rho_4 + \dot{\omega} p_{24} \rho_4, \]
\[ \dot{\rho}_3 = \rho_4 - c_{34} \dot{\theta} \rho_4 + n_{333} \rho_3^2 + n_{344} \rho_4^2 + \dot{\omega} p_{33} \rho_3, \]
\[ \dot{\rho}_4 = a_0 (\omega - \omega^*) \rho_3 + c_{43} \dot{\theta} \rho_3 + n_{434} \rho_3 \rho_4 + \dot{\omega} p_{44} \rho_4. \]
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Sinc Discretization

We use numerical techniques to analyze solutions to (6), working in one space dimension, with the saturated nonlinearity $g(s) = \frac{s^3}{1+s^2}$.

First, we compute the coefficients of (6), which depend on computing the generalized kernel of $JH_\omega$ and derived quantities.

To do so, we use a sinc-collocation method. The sinc function, $\sin(\pi x)/(\pi x)$ was used to compute solitary wave solutions when analytical expressions were not readily available in [5] and in the forthcoming [8]. It has also been used extensively in other settings.

For the purposes of our simulations, we believe we have sufficient precision, approximately ten significant digits, for the time integration of our system of ODEs.
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For the purposes of our simulations, we believe we have sufficient precision, approximately ten significant digits, for the time integration of our system of ODEs.
With the numerical schemes outlined above, we compared our finite dimensional model to the numerically integrated solution with appropriate initial data:

- In Figures 2, 3, we take $\rho_4(0) > 0$ and vary $\rho_3(0)$ and $\omega_0$, with $\theta_0 = 0$ for simplicity.
- Similarly, in Figures 4, 5, we take $\rho_4(0) < 0$ and once again vary $\rho_3(0)$ and $\omega_0$. 
Analysis

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Figure: A plot of the solution to the system of ODE’s as well as the full solution to (1) derived for solutions near the minimal soliton for $\rho_3(0) > 0$ and $\rho_3(0) < 0$, $\rho_4(0) > 0$ for $\omega_0 = .177588$, $N = 1000$. 
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Conclusions

- We find that the dynamical system (6) is an accurate approximation to the full nonlinear solution in a neighborhood of the minimal mass soliton.
- For solutions which can support soliton structure, we find that the system is oscillatory regardless of initial conditions; it is quickly attracted to the stable side of the soliton curve. If we initialize with the unstable conditions found in [3], the ODEs initially move in the unstable direction but quickly reverse.
- We cannot numerically verify our conjecture that soliton-supporting perturbations of unstable solitons dynamically select stable solitons. Instead, we see oscillatory behavior for these initial conditions, about the minimal mass soliton.
- If we begin with initial conditions which cannot support a soliton, we find that the finite dimensional dynamics move toward the value $\omega = 0$ rather quickly. Mass conservation guarantees that the full solution cannot continue along the soliton curve indefinitely.
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3. Current and Future Work
Basic Definitions

Consider the initial value problem for the generalized Korteweg-deVries equation (gKdV) in $\mathbb{R} \times \mathbb{R}^+$:

$$
\begin{cases}
  u_t + \partial_x (f(u)) + u_{xxx} = 0 \\
  u(x, 0) = u_0(x),
\end{cases}
$$

where the nonlinearity $f(s)$ is again of a saturated form.

For KdV, solitons take the form

$$u(t, x) = \phi_c(x - ct),$$

where $c > 0$ and $\phi_c(x)$ is a positive, radially symmetric, exponentially decaying solution of the equation:

$$
-c\phi_c + f(\phi_c) + \partial_{yy}\phi_c = 0.
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There is a similar well-posedness and stability theory available for (gKdV) to NLS.
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where $c > 0$ and $\phi_c(x)$ is a positive, radially symmetric, exponentially decaying solution of the equation:

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(9)

There is a similar well-posedness and stability theory available for (gKdV) to NLS.
Linear Analysis

Suppose that $u(x, 0) = \phi_c(x) + \rho_0(x)$, where $\rho_0$ is a small perturbation. If we evolve $u(x, t) = \phi_c(x - ct) + \rho(x - ct, t)$, we would like to determine the stability of the soliton by analyzing what happens to $c$ and $\nu$ over time.

Linearizing, we find that

$$\rho_t = A_c \rho$$  \hspace{1cm} (10)

where

$$A_c = \partial_y(-\partial_{yy} + c - f'(\phi_c)).$$  \hspace{1cm} (11)
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A_c = \partial_y (-\partial_{yy} + c - f'(\phi_c)).
\]
Generalized Kernel of $A_c$

The operator $A_c$ has as its continuous spectrum the entire imaginary axis. It also has an embedded eigenvalue at 0. The generalized eigenfunctions associated with the 0 eigenvalue are:

$$A_c(\partial_y \phi_c) = 0$$
$$A_c(\partial_c \phi_c) = -\partial_y \phi_c$$

Additionally, if $c^*$ is a critical point of the soliton mass curve, there is a third element $\alpha$ of the generalized kernel satisfying

$$A_{c^*}(\alpha) = \partial_c \phi_c.$$  \hspace{1cm} (12)
Linear Difficulties

Because 0 is an embedded eigenvalue in the continuous spectrum of $A_c$, and also because the extra generalized eigenvalue $\alpha$, which we are interested in, is not in $L^2$, we must work in a better spectral setting. We choose to work within the weighted spaces first developed for KdV by Pego and Weinstein.[7]

That is, we consider our soliton perturbations to be in the space $H^1_a := \{ f : e^{ay}f(y) \in H^1(\mathbb{R}) \}$. In this setting, the continuous spectrum is moved into the left half plane, leaving a neighborhood around $\lambda = 0$ in which only eigenvalues are possible in the spectrum. We will assume that 0 is the only $L^2_a$ eigenvalue of $A_c$ for $c$ near $c^*$.

In addition, for appropriate values of $a$ (near 0), we have that $\alpha \in L^2_a$, and therefore the generalized kernel of $A_{c^*}$ in $L^2_a$ is three-dimensional.
Linear Difficulties

Because 0 is an embedded eigenvalue in the continuous spectrum of $A_c$, and also because the extra generalized eigenvalue $\alpha$, which we are interested in, is not in $L^2$, we must work in a better spectral setting. We choose to work within the weighted spaces first developed for KdV by Pego and Weinstein.[7]

That is, we consider our soliton perturbations to be in the space $H^1_a := \{ f : e^{ay} f(y) \in H^1(\mathbb{R}) \}$. In this setting, the continuous spectrum is moved into the left half plane, leaving a neighborhood around $\lambda = 0$ in which only eigenvalues are possible in the spectrum. We will assume that 0 is the only $L^2_a$ eigenvalue of $A_c$ for $c$ near $c^*$.

In addition, for appropriate values of $a$ (near 0), we have that $\alpha \in L^2_a$, and therefore the generalized kernel of $A_{c^*}$ in $L^2_a$ is three-dimensional.
Ansatz

Suppose that $c^*$ is a value of $c$ at which the soliton mass curve has a critical point. Then, as with (NLS), we cannot predict the stability of the soliton on linear considerations alone. We now wish to set up our perturbed soliton in such a way as to modulate the important parameters and generate a finite dimensional system of ODEs that provides a good model of the problem.

Define $e_{3,c^*} = \alpha$ as in (12), and define $e_{3,c}$ to be a smooth extension of $e_{3,c^*}$ to nearby values of $c$. 
Then, as in [2], we make the ansatz

\[
    u(x, t) = \phi_{c_* + \eta(t)} \left( x - \int_0^t (c_* + \eta(s)) ds - \xi(t) \right) \\
    + \zeta(t)e_{3, c_* + \eta(t)} \left( x - \int_0^t (c_* + \eta(s)) ds \right) \\
    + \nu \left( x - \int_0^t (c_* + \eta(s)), t \right)
\]  

(13)

Here \( \eta, \xi, \) and \( \zeta \) will be modulated. The function \( \nu \) represents the projection onto the continuous spectrum which we hope remains small.
The Work of Comech, Cuccagna, and Pelinovsky

In [2], Comech, Cuccagna, and Pelinovsky consider a critical mass soliton for (gKdV) in the exponentially weighted spaces. They make the ansatz on the previous slide, and prove that for certain choices of the initial conditions $\zeta(0)$ and $\eta(0)$, there is a purely nonlinear instability at the minimal mass.

In this work, as with NLS, we are interested in preserving more terms in the finite dimensional system in order to prove a longer-term shadowing theorem. Ultimately we hope to resolve the long-term dynamics of small perturbations of the minimal mass soliton.
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The Finite Dimensional System

After making the ansatz in (13), and simplifying the resulting equations as much as possible, neglecting the component parallel to the continuous spectrum, and keeping terms up to quadratic order, we obtain an ODE system of the following form:

\[ \dot{\eta} - \dot{\zeta} = -A\zeta^2 \]  
\[ \dot{\zeta} - \lambda'_c \eta \zeta = -B\zeta^2, \]

(14a) \hspace{1cm} (14b)

where \( A \) and \( B \) are appropriate constants depending on the structure of the generalized kernel at \( c^* \).

Note that \( \xi \) turns out not to affect the \( (\eta, \zeta) \) dynamics, so it can be neglected, leaving us with a two-dimensional system.
Figure: The phase plane for the quadratic ODE system we obtain for KdV. The blue line of degenerate critical points represent the stable part of the soliton curve; the red line is the unstable part. The highlighted region is the basin of stability of the blue line in the simplified ODE system.
Numerical Methods

To analyze the accuracy of this system, we

- Use the sinc spectral method to generate the elements of the generalized kernel and associated quantities.
- Numerically integrate the full pde using the linearized implicit finite difference method developed by Djidjeli et. al. [4]
- Use the Matlab solver *ode45* to integrate the ode system for comparison to the numerically solved pde.
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Analysis

With the numerical schemes outlined above, we compared our finite dimensional model to the numerically integrated solution with appropriate initial data:

- In the first two slides, we take $\zeta > 0$ and vary $\eta(0)$. The first figure shows small initial conditions, while the second picture shows larger initial conditions.
- Similarly, in the latter two slides we take $\zeta < 0$ and once again vary $\eta(0)$.
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Pictures, Part 1
The Generalized Korteweg-deVries Equation

Pictures, Part 2

$p_1 = -0.1$, $p_2 = 0.01$

$p_1 = 0.01$, $p_2 = 0.1$
The Generalized Korteweg-deVries Equation Results

Pictures, Part 3

p1 = -0.001, p2 = -0.001

p1 = 0.001, p2 = -0.001

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Pictures, Part 4

\[ p_1 = -0.01, p_2 = -0.1 \]

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Conclusions

- We find that the dynamical system (14) is an accurate approximation to the full nonlinear solution in a neighborhood of the minimal mass soliton.
- We find that the underlying dynamics of KdV near the minimal mass soliton are fundamentally different than those of NLS; there is no oscillatory behavior.
- For solutions which can support soliton structure, we find that the system moves steadily toward the stable branch of the soliton curve.
- If we begin with initial conditions which cannot support a soliton, we find that the finite dimensional dynamics move away from the stable branch of the curve.
- We cannot yet numerically or analytically verify our conjecture that soliton-supporting perturbations of unstable solitons dynamically select stable solitons.
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Outline

1. The Nonlinear Schrödinger Equation
   - Soliton Stability
   - The ODE System
   - Numerical Methods
   - Results
   - Conclusions

2. The Generalized Korteweg-deVries Equation
   - Set Up
   - Results
   - Conclusions

3. Current and Future Work
We continue to work with the gKdV system.

- Our current work focuses on handling the fact that $e_3, c^*$ is not in $L^2$. This provides an obstacle to capturing the true dynamics of $H^1$ perturbations of the minimal mass soliton. This is not a problem when trying to prove an instability result, as in [2], because the solution is then being pushed away from $c^*$. However, we must confront the issue directly, using cutoff functions.
- We believe that we can prove an analytic shadowing theorem for these finite-dimensional dynamics. This would allow us to theoretically justify the phase plane which we exhibited above.
- Ultimately our goal is to return to saturated NLS and be able to prove an analytic result in that setting.
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