Landau damping and macroscopic irreversibilty for plasmas and galaxies

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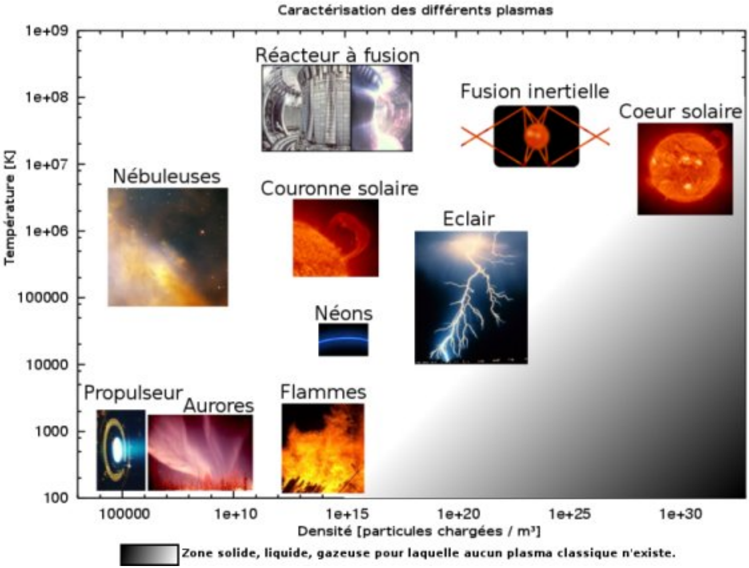
Conference SIAM-APDE
San Diego, 17th of Nov. 2011

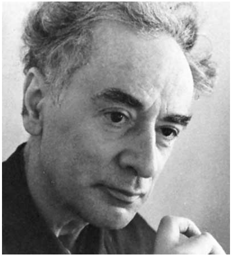
Joint work with C. Villani arXiv:0904.2760
To appear in *Acta Mathematatica*

What are plasmas?

- Totally or partially Ionized gas: ions separated from electrons
- $\rightarrow$ "Soup" of electrons with negative charges + ions with positive charges
- Electrico-magnetic fields created by plasma when these positive/negative charges not locally equilibrated
- These fields then act back on the plasma: nonlinear
- "Fourth state of matter" with its own specific properties: filamentation (cf. aurora), "boomerang-like" effects due to electric field and Langmuir waves...


Caractérisation des différents plasmas
Réacteur à fusion
Fusion inertielle
Coeur solaire
Nébuleuses
Couronne solaire
Eclair
Néons
Propulseur
Aurores
Flammes
Température [K]
1e+09
1e+08
1e+07
1e+06
100000
10000
1000
100
100000
1e+10
1e+15
1e+20
1e+25
1e+30
Densité [particules chargées / m³]
Zone solide, liquide, gazeuse pour laquelle aucun plasma classique n'existe.

Lev Landau



- Lev Davidovitch Landau, russian physicist 1908-1968
- Numerous fundamental contributions

The discovery of the “damping”

- Two major contributions in plasma physics
- 1936: how to modify the Boltzmann equation (describing collisional rarefied gases) for collisions in plasmas $\rightarrow$ Landau-Coulomb equation
- Vlasov 1938: no collision on timescale of observation of plasmas, interactions due to mean-fields $\rightarrow$ equation of Vlasov-Poisson in the non-relativistic non-magnetic case
- 1946: Landau suggests by a clever computation on the linearized Vlasov-Poisson around homogeneous equilibria damping of spatial waves, hence reorganization of the plasma along time to electric neutrality and electric field decays
- Landau even computes a rate of decay (exponential)

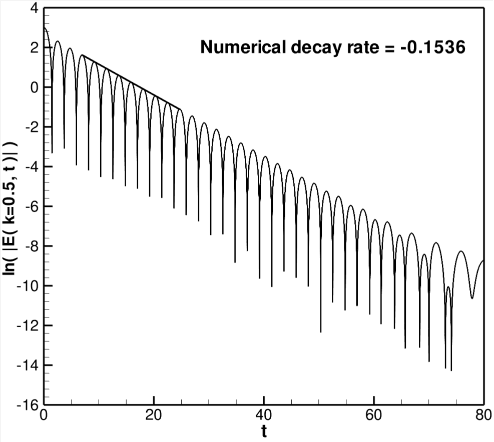
A surprising discovery

- In the theory of Vlasov the electrons of the plasma interact through their collective mean electric field, without collisions
- Then why the prediction of damping is so surprising? Because unlike collisional kinetic equations, the mean-field description of interaction yields reversible equation
- The equation is unchanged when reversing time and velocities $(t, x, v) \to (-t, x, -v)$ (as for the Newton equations)
- The 1946 paper of Landau is the starting point of more than half a century of debates among physicists: observed experimentally but controversies on the nonlinear validity, and on the explanation of the mechanism

Controversies and difficulties

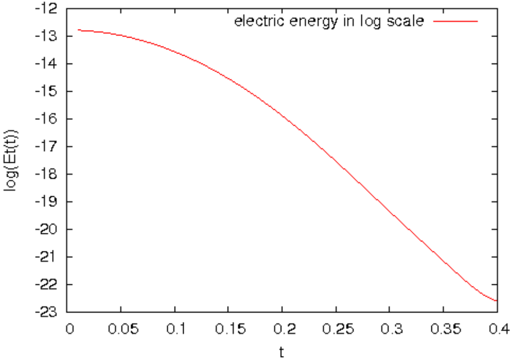
- Reversible evolution equation and irreversible phenomenon (electric field goes to zero)
- Landau computation linear: nonlinear instability?
- Filamentation seems contradictory with spatial homogeneization and yields velocity oscillations
- Hard to draw definitive conclusions from numerical simulations since genuinely nonlinear effects only for long times and filamentation (= oscillations) produces small scales
- Lynden-Bell 1962 proposes a similar mechanism for galactic dynamics ("gas of stars")

Numerics (I) (F. Filbet, Univ. Lyon)



Electric field (1-dim, repulsive, perturbation of Maxwellian)

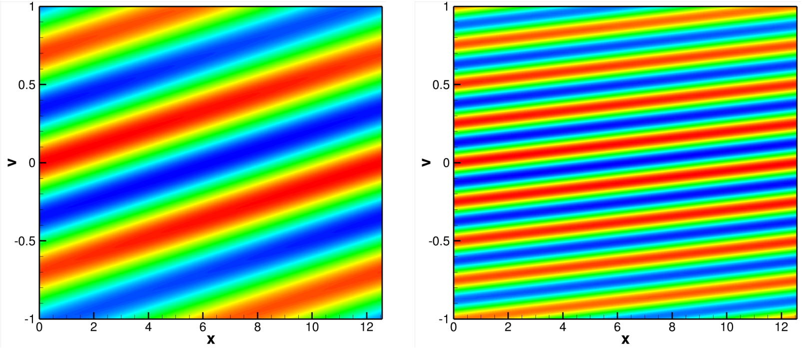
Numerics (II) (F. Filbet, Univ. Lyon)



Gravitational field (1-dim, attractive, perturbation of Maxwellian)

Numerics (III) (I. Gamba and P. Morrison, UT Austin)

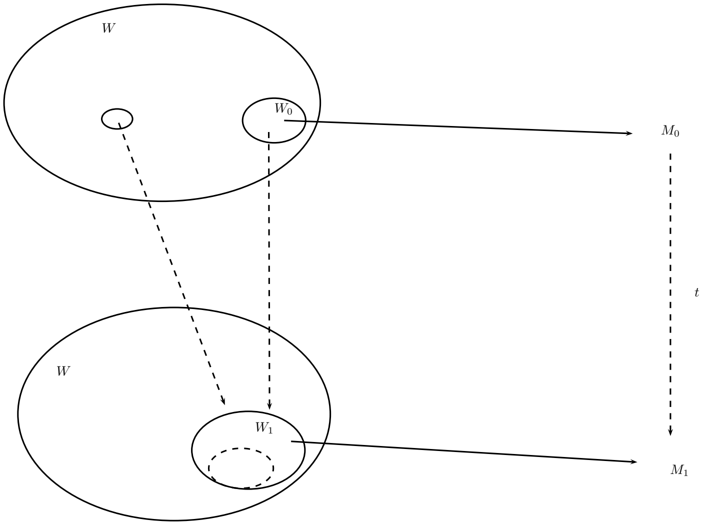
Filamentation (repulsive case)...



Boltzmann, entropy and irreversibility

- Maxwell 1867 discovers the Boltzmann equation and the velocity distribution of a gas at equilibrium (gaussian)
- Boltzmann 1872: "*it has yet not been proved that, for any initial state of the gas, it has to approach the limit discovered by Maxwell*"
- To solve this question, Boltzmann proposes a formal proof of the increase of entropy (H-theorem) together with an interpretation of irreversibility

$S = k \log W$, irreversibility and "molecular chaos"



$S = k \log W$

The Boltzmann equation (1867-1872)

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \partial_x f}_{\text{space change}} = \underbrace{Q(f,f)}_{\text{collision}} \quad \text{on } f(t,x,v) \geq 0$$

- Transport term $v \cdot \partial_x$: straight line with velocity v (inertia principle)
- Collision operator $Q(f,f)$: bilinear, acts on v only, integral (non-local)

$$Q(f,f)(v) = \int_{v_*} \int_{collisions} \Big[\underbrace{f(v')f(v'_*)}_{(v',v'_*)\to(v,v_*)} - \underbrace{f(v)f(v_*)}_{(v,v_*)\to\ldots} \Big] B$$

The Vlasov-Poisson equation (1938)

Landau-Coulomb equation 1936:

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \partial_x f}_{\text{space change}} = \underbrace{Q(f,f)}_{\text{Landau collision operator}}$$

However plasmas and galaxies non-collisional at usual timescales
Vlasov-Poisson equation 1938:

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \partial_x f}_{\text{straight line along } v} - \underbrace{E \cdot \partial_v f}_{\text{velocity change (force } E)} = 0$$

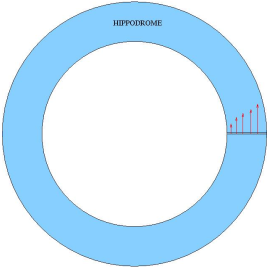
Mean-field interaction (electric, gravitation)

$$E = \nabla_x(\phi * \rho) \quad \rho[f] = \int_v f$$

Landau damping: $E \to 0$ for perturbations of stable steady states

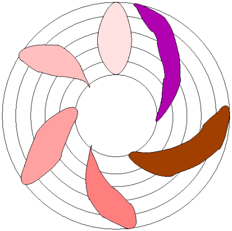
Informal ideas of the strategy: phase mixing

- Our starting point $\neq$ dominant interpretation among physicists, rather interpretation of the 1950s by phase mixing



- If velocity "different enough" ($\mathbb{Q}$-free) distribution of runners shall be quite mixing at first sight
- Mixing phenomenon at the basis of filamentation

- Cf. ergodic dynamical systems
- Irreversible behavior *on the average* out of a reversible dynamics on each particle...
- But in this example this homogeneization only occurs **when averaged in time**: cf. Poincaré's recurrence theorem
- Transport equation $\partial_t f + v \cdot \partial_x f = 0$ in the torus $x \in \mathbb{T}^d$ corresponds roughly to infinite continuum number of runners



- In this limit, no recurrence and no more need of time averages

Convergence by phase mixing

- The phase mixing described above implies a weak convergence $f \rightharpoonup f_{\pm\infty}$ and a strong convergence (homogeneization) of the spatial density

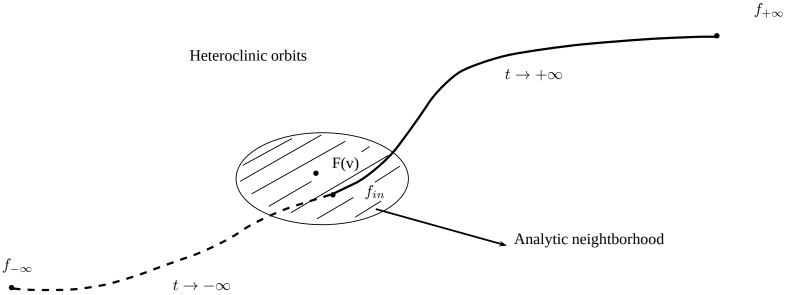
$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv \to \rho_\infty$$

  $\Rightarrow$ damping of the electric field $E = E[\rho] \to 0$
- This relaxation to equilibrium of some partial average is compatible with reversibility of the equation (think to $u(t,x) = e^{itx} u_0(x)$)
- But so far we have not considered interactions between particules (free transport)
- Question: what happens when one adds the nonlinear electric interactions?

A perturbative approach

- Our work can be summarized in the following answer (under assumptions of regularity and perturbation): phase mixing survives **for all times** to the nonlinear mean-field interactions of the Vlasov-Poisson equation, which implies the damping
- This mathematical theorem yields back some physics answers:
  - The damping does not happen *thanks* to interaction but *in spite of them*
  - Filamentation actually helps instead of being a problem
  - Linear PDE Hamiltonian mechanism which survives a complicated nonlinear perturbation **for all times**
  - Through this perturbation its structure also becomes **richer** (cf. limit states, heterclinic orbits. . .)

Illustration



Cf. Idea of Boltzmann of irreversibility: "microscopic" state f and (observable) "macroscopic" state $\rho = \rho[f]$

However major difference: no closed equation on ρ

It keeps memory of the history of the whole (f) system: equilibration without increase of entropy

Note also that $f_{+\infty}$ and $f_{-\infty}$ are in the neighborhood of f^0!

An important inspiration...

- 1954: Kolmogorov "KAM" theorem (Arnold-Moser)
- Issue of stability of solar system: OK when neglecting planets interactions
- Perturbation of such "completely integrable" systems remain stable "for all times" (for majority of trajectories)
- Conceptual inspiration (however different): "completely integrable" system is the *linearized Vlasov-Poisson equation*
- Abstract Newton scheme at the PDE level global in time in analytic space

Difficulties and. . . surprises

- But why is it possible to control the nonlinear perturbation?
- Gliding functional spaces following filamentation along free transport semigroup: time-dependent norms and associated functional analysis
- Finite-time deflection estimates on trajectories sharing common flavor with Nekoroshev theorem
- New mechanisms of extorsion of regularity in order to deal with composition along characteristics
- Estimates taking advantage of delay and time-localization of the nonlinearity: plasma echoes

Existing results

- Linear LD: Landau, Saenz, Degond, Maslov-Fedoryuk...
- Orbital stability (different issue but related): many recent works (Wolansky, Guo, Lin, Rein, Lemou-Mehats-Raphaël...)
- Glassey-Schaeffer: Obstruction to LD without confinement
- Numerics: Zhou-Guo-Shu, Heath-Gamba-Morrison-Michler...
- Lin-Zeng: Obstruction to LD for low regularity
- Nonlinear LD: Caglioti-Maffei, Hwang-Velazquez: existence of some damped solutions but no information about which and how many initial data would yield a damped solution...
- Short list but essentially exhaustive for mathematical studies

The linear damping

$$(LVP) \quad \boxed{\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f^0(v) = 0}, \quad x \in \mathbb{T}^d_L,\ v \in \mathbb{R}^d.$$

Theorem (CM-Villani)

Assume one of the following conditions:

(1) $\left(\max_{k \neq 0} |\hat{\phi}(k)|\right) \left(\sup_{|\sigma|=1} \int_0^{+\infty} \left|\tilde{f}^0(r\sigma)\right| \, r \, dr\right) < \frac{1}{4\pi^2}$

Covers Newton interaction below Jeans scale

(2) Or $\hat{\phi} \geq 0$ and $z\, F'(z) < 0$ for any one-dimensional marginal F of f^0: covers Coulomb interaction at all scales

Then $\rho \to c$ (average) and $f \rightharpoonup f_\infty(v)$ (space average). Rate dictated by regularity (can be relaxed at least down to Sobolev).

Non-linear Landau damping

$$(VP) \qquad \boxed{\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f = 0} \qquad x \in \mathbb{T}^d_L, \; v \in \mathbb{R}^d$$

Theorem (CM-Villani)

Assume (i) $f^0(v)$ and ϕ satisfy the linear damping condition;

(i) ϕ is not too singular: $\left|\hat{\phi}(k)\right| \leq \frac{C}{|k|^{1+\gamma}}$, $\gamma \geq 1$

(**Coulomb/Newton is included with** $\gamma = 1$);

(ii) $\delta := \|f_{in} - f^0\|_* \leq \varepsilon << 1$: exponential localization in v, k, η.

Then (a) $\rho[f](t,x) \xrightarrow{t \to \pm\infty} c$ strongly (exp. fast);

(b) $f(t,x,v) \rightharpoonup f_{\pm\infty}(v)$ as $t \to \pm\infty$ weakly;

(c) $\langle f \rangle(t,v) \xrightarrow{t \to \pm\infty} f_{\pm\infty}(v)$ strongly (space average).

Generalization to Gevrey spaces (allows compactly supported perturbations) with rate like $e^{-ct^{\alpha}}$, $\alpha \in (0,1)\dots$

What do we learn from this result?

- f_t converges for all time without appealing to extra randomness
- $f(t, x + vt, v)$ remains close to $f^0(v)$ in analytic norms (analytical orbital stability along free transport);
- limit not determined by conservation laws or thermodynamical issues: in general $f_{-\infty} \neq f_{+\infty}$ (memory)
- convergence "for no reason" just because near the "completely integrable" linear case: reminiscent of KAM
- A whole neighborhood (in analytic topology) of f^0 is filled with homoclinic/heteroclinic (in weak topology) trajectories
- From the proof: constructive scheme to approximate the whole dynamics and the limits $f_{\pm\infty}(v)$ in terms of δ

Basic fundamental tool: Fourier transform

Consider $f = f(x, v)$ where $x \in \mathbb{T}_L^d := \mathbb{R}^d/(L\mathbb{Z}^d)$ and $v \in \mathbb{R}^d$

We denote

$$\hat{f}(k, v) := \int_{\mathbb{T}_L^d} e^{-2i\pi \frac{k}{L}\cdot x} f(x, v)\, dx, \qquad k \in \mathbb{Z}^d$$

$$\tilde{f}(k, \eta) := \int_{\mathbb{T}_L^d \times \mathbb{R}^d} e^{-2i\pi \frac{k}{L}\cdot x}\, e^{-2i\pi \eta\cdot v} f(x, v)\, dx\, dv, \qquad k \in \mathbb{Z}^d,\ \eta \in \mathbb{R}^d$$

W.l.o.g. set $L = 1$ in the sequel. . .

Phase mixing for the free transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0 \text{ in } \mathbb{T}^d \times \mathbb{R}^d \text{ with initial data } f_i(x, v)$$

- mass preserved and explicit solution $f(t, x, v) = f_i(x - vt, v)$
- zero spatial mode ($k = 0$) is preserved:

$$\forall\, v \in \mathbb{R}^d, \quad \int_{\mathbb{T}^d} f(t, x, v)\, dx = \int_{\mathbb{T}^d} f_i(x, v)\, dx$$

- Fourier transform:

$$\boxed{\tilde{f}(t, k, \eta) = \tilde{f}_i(k, \eta + kt)} \quad \text{and} \quad \boxed{\tilde{\rho}(t, k) = \tilde{f}_i(k, kt)}$$

- Hence convergence to zero for any $k \neq 0$, with a rate given by the smoothness of f_i in v
- Phase mixing estimate: $|f(t, k, \eta)| \leq C\, e^{-\lambda|\eta + kt|}$ (analyticity)

The linearized Vlasov equation (I)

Consider the VP equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - E[f] \cdot \nabla_v f = 0, \quad x \in \mathbb{T}^d,\ v \in \mathbb{R}^d$$

Linearize around a spatially homogeneous profile $f^0 = f^0(v)$:
$E[f^0] = 0$ and therefore:

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla_x f - E[f] \cdot \nabla_v f^0 = 0}$$

Recall that

$$E[f] = (\nabla_x \phi) * \rho[f]$$

where

$$\rho[f](t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$$

The linearized Vlasov equation (II)

Duhamel formulation along free transport:

$$f(t,x,v) = f_i(x-vt,v) + \int_0^t (\nabla\phi * \rho)(s, x-v(t-s)) \cdot \nabla_v f^0(v)\, ds$$

Integrate in v and Fourier transform in x:

$$\hat{\rho}(t,k) = \tilde{f}_i(k,kt) - 4\pi^2 \hat{\phi}(k) \int_0^t \hat{\rho}(s,k)\, \tilde{f}^0(k(t-s))\, |k|^2\, (t-s)\, ds$$

$\rightarrow$ Volterra equations decoupled for each mode k for the density:

$$\boxed{\varphi(t) = a_k(t) + \int_0^t K_k(t-s)\, \varphi(s)\, ds}$$

An elementary computation (assumption (1))

Assumption (1) means for the kernel K_k of the Volterra equation of any k mode: For some $\lambda > 0$:

$$\int_0^{+\infty} e^{2\pi\lambda s}\, K_k(s)\, ds < 1 - \kappa, \qquad \kappa \in (0,1)$$

Then by simple computations one gets

$$\int_0^{+\infty} e^{2\pi\lambda t}\, \varphi(t)\, dt \leq \frac{\int_0^{+\infty} e^{2\pi\lambda t}\, a_k(t)\, dt}{1 - \int_0^{+\infty} e^{2\pi\lambda s}\, K_k(s)\, ds} \leq \frac{1}{\kappa} \int_0^{+\infty} e^{2\pi\lambda t}\, a_k(t)\, dt$$

Therefore exponential decay of the modes of the density in terms of the initial regularity in v (recall $a_k = \tilde{f}_i(k, kt)$)

Analysis of a Volterra equation (assumption (2))

Let us first discuss informally:

$$\varphi(t) = a_k(t) + \int_0^t K_k(t-s)\,\varphi(s)\,ds$$

Laplace transform: $\boxed{\varphi^L = a_k^L + K_k^L\,\varphi^L}$
Hence formally φ should be given by

$$\varphi = (\text{Laplace})^{-1}\left(\frac{a^L}{1-K^L}\right)$$

We expect the correct condition is: K^L does not approach 1 in a strip including some $\{0 < \Re\xi < \lambda\}$ in the complex plane

Analysis of a Volterra equation (assumption (2))

More precise statement:

$$\varphi(t) = a(t) + \int_0^t K(t-s)\,\varphi(s)\,ds$$

(i) Assume $K(t) = O(e^{-2\pi\lambda t})$

(ii) Define $\mathcal{L}(\xi) := \int_0^{+\infty} e^{2\pi\xi^* t}\,K(t)\,dt$ and assume (for some $\epsilon > 0$)

$$(L) \qquad \forall\,\xi \in \{-\epsilon < \Re\xi < \lambda\}, \quad |\mathcal{L}(\xi) - 1| \geq \kappa > 0$$

Then for all $\lambda' < \lambda$:

$$\sup_{t\geq 0} |\varphi(t)| e^{2\pi\lambda' t} \leq C(\lambda, \lambda', \kappa) \sup_{t\geq 0} \Big(|a(t)| e^{2\pi\lambda t}\Big)$$

Application to the case of assumption (2)

It remains to prove the following proposition (technical):
The condition

$$(L) \qquad \forall\, \xi \in \{-\epsilon < \Re\xi < \lambda\}, \quad |\mathcal{L}(\xi) - 1| \geq \kappa > 0$$

is satisfied as soon as

(2) $\hat{\phi} \geq 0$ and $z\, F'(z) < 0$ for any one-dimensional marginal F of f^0 (good for Coulomb interaction at all scales)

Specifications for the norms

- Analyticity of f compulsory in order to get exponential decay, remark however that at the linear level any regularity would be ok (price: slower decay)
- Analytic norms based on Fourier seem well-adapted in the x variable in view of the previous linearized computation (discrete, decoupling)
- Analytic estimate (in v) of the characteristics: not L^1/L^2 in v $\rightarrow$ analytic norms in v "of L^∞" type
- The "gliding regularity" means that we should include a time-shift in the definition of the norm, accounting for the "cascade" of free transport

Analytic norms in one variable

Two natural families of analytic norms:

$\mathcal{C}^{\lambda;p}$ norm: $\sum_{n \in \mathbb{N}^d} \frac{\lambda^n}{n!} \|\nabla^n f\|_{L^p}$

$\mathcal{F}^\lambda$ norm: $\sum_{k \in \mathbb{Z}^d} e^{2\pi|k|} |\hat{f}(k)|$

$\mathcal{F}^\lambda$ and $\mathcal{C}^{\lambda;\infty}$ are algebra and as a consequence will satisfy nice composition properties.

Example:

$$\|f \circ (\mathrm{Id} + G)\|_{\mathcal{Y}^\lambda} \leq C \|f\|_{\mathcal{Y}^{\lambda+\nu}}$$

where $\nu = \|G\|_{\mathcal{Y}^\lambda}$

$\rightarrow$ strongly non-linear behavior of composition (compare to, say, Sobolev norms)

Analytic norms in two variables

Of course one could define $\mathcal{C}^{\lambda;p}$ and $\mathcal{F}^{\lambda}$ with two variables. But, from the specifications discussion we shall hydridize them:

$$\|f\|_{\mathcal{Z}^{\lambda,\mu;p}} := \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}^d}\frac{\lambda^n}{n!}e^{2\pi\mu|k|}\|\nabla_v^n\hat{f}(k,v)\|_{L^p(dv)}$$

Remark: In the proof we need in fact even more flexibility, with an additionnal index for a Sobolev correction in the x variable:

$$\|f\|_{\mathcal{Z}^{\lambda,(\mu,\gamma);p}} := \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}^d}(1+|k|)^{\gamma}\frac{\lambda^n}{n!}e^{2\pi\mu|k|}\|\nabla_v^n\hat{f}(k,v)\|_{L^p(dv)}$$

For $p=\infty$, still an algebra

Gliding norms (I)

Now we introduce the time-shift, the guiding principle is (remember the discussion on the free transport):

$$\|f\|_{\mathcal{Y}^{**}_\tau} = \|f \circ S^0_\tau\|_{\mathcal{Y}^{**}_0} \qquad \text{(and thus also } \|f\|_{\mathcal{Y}^{**}_{t+\tau}} = \|f \circ S^0_\tau\|_{\mathcal{Y}^{**}_t}\text{)}$$

where $S^0_\tau(x, v) = (x + \tau v, v)$ characteristics semigroup of free tspt

Therefore we obtain

$$\|f\|_{\mathcal{Z}^{\lambda,(\mu,\gamma);p}_\tau} := \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}^d} (1 + |k|)^\gamma \frac{\lambda^n}{n!} e^{2\pi\mu|k|} \|(\nabla_v + 2i\pi\tau k)^n \hat{f}(k, v)\|_{L^p(dv)}$$

For $p = \infty$, still an algebra

Gliding norms (II)

More intricate composition properties, e.g.

$$\|f(x+bv+X, av+V)\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} \le |a|^{-d/p} \|f\|_{\mathcal{Z}_\sigma^{\alpha,(\beta,\gamma);p}}$$

with (!)

$$\alpha = \lambda|a| + \|V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}, \qquad \beta = \mu + \lambda|b+\tau-a\sigma| + \|X-\sigma V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}$$

(crucial for treating characteristics)

Finally it can be proved (tedious) some comparison results with more "usual norms" such as:

$$\|f\|_{\lambda,\mu,\beta} = \sup_{k,\eta}\Big(|\tilde{f}(k,\eta)|\, e^{2\pi\lambda|\eta|}\, e^{2\pi\mu|k|}\Big) + \iint_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|\, e^{2\pi\beta|v|}\, dv\, dx.$$

The linearized theorem reframed

Assume $\|\nabla\phi\|_{L^1} \leq C_W$, and $f_i(x,v)$ such that

(i) Laplace transform condition of the linearized study holds for some constants $\lambda, \kappa > 0$

(ii) $\|f^0\|_{\mathcal{C}^{\lambda;1}} \leq C_0$

(iii) $\|f_i\|_{\mathcal{Z}^{\lambda,\mu;1}} \leq \delta$ for some $\mu > 0$

Then for any $\lambda' < \lambda$, $\mu' < \mu$,

$$\sup_{t\in\mathbb{R}} \left\|f(t,\cdot)\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}} \leq C\,\delta \quad \text{and} \quad \sup_{t\in\mathbb{R}} \left\|\rho(t,\cdot)\right\|_{\mathcal{F}^{\lambda'|t|+\mu'}} \leq C\,\delta$$

for some constant $C = C(d, C_W, C_0, \lambda, \lambda', \mu, \mu', \kappa)$

Ideas of the proof

- First estimate

$$\sup_{t\geq 0}\left\|\rho(t,\cdot)\right\|_{\mathcal{F}^{\lambda' t+\mu'}} = \sup_{t\geq 0}\sum_{k\in\mathbb{Z}^d} e^{2\pi(\lambda' t+\mu')|k|}|\hat{\rho}(t,k)|$$

  by summing the estimate we have for each mode and using the margin on μ to get k-summability

- Then estimate

$$\sup_{t\in\mathbb{R}}\left\|f(t,\cdot)\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}}$$

  using the Hölder like inequality

$$\left\|fg\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}} \leq \left\|f\right\|_{\mathcal{Z}_t^{\lambda',\mu';\infty}} \times \left\|g\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}}$$

  and the fact that $\mathcal{Z}_t^{\lambda',\mu';\infty} = \mathcal{F}^{\lambda'|t|+\mu'}$ for functions only of x

Non-linear Landau damping

$$(NLV) \qquad \boxed{\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f = 0} \quad x \in \mathbb{T}^d,\ v \in \mathbb{R}^d$$

Theorem (CM-Villani, 2009)

Assume (i) $f^0(v)$ and ϕ satisfy the linear damping conditions;

(i) ϕ is not too singular: $\left|\hat{\phi}(k)\right| \leq \frac{C}{|k|^{1+\gamma}}$, $\gamma \geq 1$;

(ii) $\delta := |||f_i - f^0||| \leq \varepsilon << 1$: exponential localization in v, k, η.

Then (a) $\rho[f](t,x) \xrightarrow{t \to \pm\infty} c$ strongly (exp. fast);

(b) $f(t,x,v) \rightharpoonup f_{\pm\infty}(v)$ as $t \to \pm\infty$ weakly;

(c) $\langle f \rangle(t,v) \xrightarrow{t \to \pm\infty} f_{\pm\infty}(v)$ strongly (space average).

Abstract Newton scheme (I)

Perturb $\partial_t f = P(f)$ around stationary solution f^0

Write Cauchy problem with initial datum $f_i \simeq f^0$ as:

$$\Phi(f) := \Big(\partial_t f - P(f),\ f(0,\cdot)\Big) = (0, f_i).$$

Newton iteration: start from f^0 and solve inductively $\Phi(f^{n-1}) + \Phi'(f^{n-1}) \cdot (f^n - f^{n-1}) = 0$ for $n \geq 1$:

$$\partial_t h^1 = DP(f^0) \cdot h^1 \quad h^1(0,\cdot) = f_i - f^0$$

$$\forall n \geq 1, \quad \partial_t h^{n+1} = DP(f^n) \cdot h^{n+1} + \big[P(f^n) - \partial_t f^n\big], \quad h^{n+1}(0,\cdot) = 0.$$

By substraction, for $n \geq 1$ this is the same as

$$\partial_t h^{n+1} = DP(f^n) \cdot h^{n+1} + \Big[P(f^{n-1} + h^n) - P(f^{n-1}) - DP(f^{n-1}) \cdot h^n\Big]$$

Abstract Newton scheme (II)

$$\partial_t h^{n+1} = \mathbf{DP}(\mathbf{f^n}) \cdot \mathbf{h^{n+1}} + \left[P(f^{n-1} + h^n) - P(f^{n-1}) - DP(f^{n-1}) \cdot h^n \right]$$

$$h^{n+1}(0, \cdot) = 0$$

- **Bold**: linearized semi-group around solution of the previous step $\rightarrow$ new difficulties (filamentation)
- Red: Source term quadratic in terms of the previous step if "P twice differentiable"
- Blue: Apart from the first step, zero initial data
- So it replaces a nonlinear problem by an infinite system of linearized problems around non-homogeneous solutions and with source term
- Formally **one hopes for a convergence like** δ^{2^n}, typical of a Newton scheme

Concrete Newton scheme for our problem

$$f^n = f^0 + h^1 + \ldots + h^n,$$

$\underline{n = 0}$:

$$\partial_t h^1 + v \cdot \nabla_x h^1 - E[h^1] \cdot \nabla_v f^0 = 0, \quad h^1(0, \cdot) = \underline{f_i - f^0}$$

$\underline{n \geq 1}$:

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + \mathbf{F}[\mathbf{f^n}] \cdot \nabla_{\mathbf{v}} \mathbf{h^{n+1}} - E[h^{n+1}] \cdot \nabla_v f^n = E[h^n] \cdot \nabla_v h^n$$

$$h^{n+1}(0, \cdot) = 0$$

In Nash-Moser approach: regularization operators at each step. Here no, but summable losses of regularity (cf. Kolmogorov proof of KAM analytic)

New issues to be solved

- Initial data enters through the first step only in the scheme, and the smallness δ should propagate
- Be careful that each step h^n is global in time
- First step is exactly the linearized study, hence the linearized stability condition (L)
- For $n \geq 1$, 3 new difficulties:

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + \mathbf{F}[\mathbf{f^n}] \cdot \nabla_{\mathbf{v}} \mathbf{h^{n+1}} - E[h^{n+1}] \cdot \nabla_v f^n = E[h^n] \cdot \nabla_v h^n$$

1. **Bold**: Perturbation of the free transport characteristics by $E[f^n]$ (force field small but not going to 0 as $n \to \infty$)
2. Blue: The background in the reaction term is now depending on t, x, v $\to$ growth of its ∇_v (filamentation)
3. Red: Quadratic source term, which should yield a quadratic error in some norm if we want the scheme to converge very fast

General strategy (I)

We propagate a number of estimates along the scheme; the most important are (slightly simplifying)

$$\sup_{\tau\geq 0}\left\|\int_{\mathbb{R}^d} h^n\big(\tau,\,\cdot\,,v\big)\,dv\right\|_{\mathcal{F}^{\lambda_n\tau+\mu_n}}\leq \delta_n,$$

$$\sup_{t\geq\tau\geq 0}\left\|h^n\big(\tau,\Omega^n_{t,\tau}\big)\right\|_{\mathcal{Z}^{\lambda_n(1+b),\mu_n;1}_{\tau-\frac{bt}{1+b}}}\leq \delta_n,\qquad b=b(t)=\frac{B}{1+t},$$

$$\left\|\Omega^n_{t,\tau}-\mathrm{Id}\right\|_{\mathcal{Z}^{\lambda_n(1+b),(\mu_n,\gamma);\infty}_{\tau-\frac{bt}{1+b}}}\leq C\,\left(\sum_{k=1}^{n}\frac{\delta_k\,e^{-2\pi(\lambda_k-\lambda_{n+1})\tau}}{2\pi(\lambda_k-\lambda_{n+1})^2}\right)\min\{t-\tau\,;\,1\}.$$

$\Omega^n_{t,\tau}$ denote the (finite-time) deflection operators, see below

Additional difficulty: Stratification of all the estimates (multiscale)

General strategy (II)

- **Step 1.** estimate $\Omega^n - \mathrm{Id}$ (the bound should be uniform in n);
- **Step 2.** estimate $\Omega^n - \Omega^k$ ($k \leq n-1$; the error should be small when $k \to \infty$);
- **Step 3.** estimate $\nabla\Omega^n - \mathrm{Id}$;
- **Step 4.** estimate $(\Omega^k)^{-1} \circ \Omega^n$ ($1 \leq k \leq n$);
- **Step 5.** estimate h^k ($1 \leq k \leq n$) and its derivatives along the composition by Ω^n;
- **Step 6.** estimate $\rho[h^{n+1}]$;
- **Step 7.** estimate $E[h^{n+1}]$ from $\rho[h^{n+1}]$;
- **Step 8.** estimate $h^{n+1} \circ \Omega^n$;
- **Step 9.** estimate derivatives of h^{n+1} composed with Ω^n;
- **Step 10.** show that for h^{n+1}, ∇ and composition by Ω^n asymptotically commute.

Implication of the convergence of the scheme

The goal is to prove sthg like

$$\forall\, t \geq 0, \quad \|h_n(t,\cdot)\|_{\mathcal{Z}_t^{\lambda_n,\mu_n}} \leq \delta_n$$

where δ_n converges very fast to 0 (summable) and λ_n decaying to $\lambda_\infty > 0$, μ_n decaying to $\mu_\infty > 0$
Then we deduce by summation that $f^n = f^0 + h^1 + \ldots + h^n \to f^\infty$ with

$$\forall\, t \geq 0, \quad \|f^\infty(t,\cdot)\|_{\mathcal{Z}_t^{\lambda_\infty,\mu_\infty}} \leq C\,\delta$$

This concludes the proof of the theorem as in the (reframed) linearized case.
Also it allows for rigorous expansion of the solution, and therefore its limiting profiles, by computing the solution to a finite number of linearized problems.

Remark on the short-time estimate

As a preliminary, in our proof, we shall need small times estimates

Cauchy-Kowalevskaya style problem (loss of one derivative)

To compensate for the loss of one derivative, allow for a loss of regularity:

$$\lambda(t) = \lambda - Kt, \quad \mu(t) = \mu - Kt$$

and use

$$\frac{d}{dt}^{+}\bigg|_{t=\tau} \|f\|_{\mathcal{Z}_\tau^{\lambda(t),\mu(t);p}} \leq -\frac{K}{1+\tau} \|\nabla f\|_{\mathcal{Z}_\tau^{\lambda(\tau),\mu(\tau);p}}$$

The characteristics method

Linearize around $\bar{f}$:

$$\partial_t f + v \cdot \nabla_x f - E[f] \cdot \nabla_v \bar{f} - E[\bar{f}] \cdot \nabla_v f = 0$$

We want to get rid of the last term by characteristics method:
$(X, V)_{s,t}(x, v)$ position/velocity at time t, starting at time s from (x, v), driven by $E[\bar{f}]$ (non autonomous):

$$\frac{dX}{dt} = V, \qquad \frac{dV}{dt} = -E[\bar{f}](X)$$

Hence: $\partial_t f(t, X_{0,t}, V_{0,t}) = \partial_t f_| + v \cdot \nabla_x f_| - E[\bar{f}] \cdot \nabla_v f_|$ and so

$$f(t, x, v) = f_i(X_{t,0}(x, v), V_{t,0}(x, v)) + \int_0^t E[f](X_{t,s}(x, v)) \cdot \nabla_v \bar{f}(s, X_{t,s}(x, v), V_{t,s}(x, v))\, ds$$

Finite-time deflection estimates

- Assume $E[\bar{f}] = \nabla\phi * \bar{\rho}$, with $\|\bar{\rho}_t\|_{\mathcal{F}^{\lambda|t|+\mu}} \leq C$
- $S_{s,t} = (X_{s,t}, V_{s,t})$: characteristics induced by E
- $S^0_{s,t} = (x + v(t-s), v)$: characteristics of free transport
- $\Omega_{t,s} = S_{t,s} \circ S^0_{s,t}$ "deflection operators" (not semigroup)
- the kind of estimates we establish:

$$\|\Omega_{t,s} - \mathrm{Id}\|_{\mathcal{Z}_{s'}^{\lambda',\mu'}} \leq C' \min\{(t-s), 1\}\, e^{-\alpha s}$$

- **Uniform in** $t >> s$**, small for** $s \sim t$ **and** $s \to +\infty$
- Proof: Fixed point theorems combined with algebra-like properties of $\mathcal{Z}$ spaces...

The bilinear term to be estimated

$$\sigma(t,x) = \int_0^t \int_{\mathbb{R}^d} \big(E[f] \cdot \nabla_v \bar{f}\big)\big(\tau, x - v(t-\tau), v\big)\, dv\, d\tau.$$

This quantity can be interpreted as follows:

If particles distributed according to f exert a force on particles distributed according to $\bar{f}$, then σ is the variation of density $\int f\, dv$ caused by the *reaction* of $\bar{f}$ on f

Regularity extorsion in short times (I)

- Straight trajectories must be replaced by characteristics (this reflects the fact that $\bar{f}$ also exerts a force on f)
- $\rightarrow$ source of considerable technical difficulties
- However, keeping in mind our finite time "scattering" estimates, one sees that a nice "bootstrap" phenomenon occurs thanks to the structure of the Vlasov-Poisson equation:
  (1) a distribution bounded in our gliding norm $\Rightarrow$ a (exponentially) damped force field
  (2) a damped force field (in the sense of our gliding norms) will create characteristics asymptotically close to free transport characteristics, with small error in these very gliding norms
- To close estimates we therefore need some "margin" at least for short times in these gliding norms

Regularity extorsion in short times (II)

- Key tool (together with finite-time deflection estimates) used to overcome them: regularity extorsion in the short time (somehow reminiscent of velocity-averaging lemmas)
- Here is a simplified version (with $b = B/(1+t)$):

$$\|\sigma(t,\cdot)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} \leq \int_0^t \left\| E\left[f(\tau,\cdot)\right]\right\|_{\mathcal{F}^{\lambda[\tau - b(t-\tau)]+\mu,\gamma}} \times \left\|\nabla f(\tau,\cdot)\right\|_{\mathcal{Z}^{\lambda(1+b),(\mu,0);1}_{\tau - bt/(1+b)}} \, d\tau.$$

- Observe that the regularity of σ is better than that of $E[f]$, with a gain that degenerates as $\tau \to t$ (and bounded in t)

Regularity extorsion in long times

We show that if $\bar{f}$ has a high gliding regularity, then the decay of σ in large time is better than what would be expected:

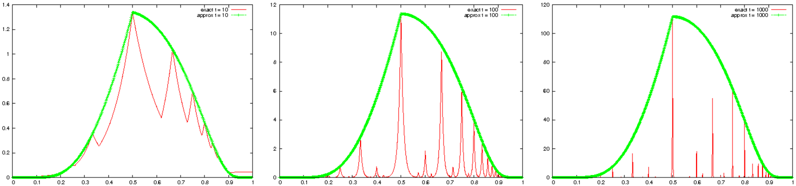
$$\|\sigma(t,\cdot)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} \leq \int_0^t K(t,\tau)\,\big\|E\big[f(\tau,\cdot)\big]\big\|_{\mathcal{F}^{\lambda\tau+\mu,\gamma}}\,d\tau,$$

where

$$K(t,\tau) = \left[\sup_{0\leq s\leq t}\left(\frac{\big\|\nabla_v \bar{f}(s,\cdot)\big\|_{\mathcal{Z}_s^{\bar{\lambda},\bar{\mu};1}}}{1+s}\right)\right] K_0(t,\tau)$$

$$K_0(t,\tau) = (1+\tau)\sup_{k\neq 0,\ \ell\neq 0}\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|}\,e^{-2\pi(\bar{\mu}-\mu)|\ell|}}{1+|k-\ell|^{\gamma}}$$

The time-response kernel K_0



The kernel $K_0(t, \tau)$, together with an approximate upper bound for $\alpha = 0.5$ and $t = 10$, $t = 100$, $t = 1000$.

Stabilization by echoes (I)

- Kernel $K(t,\tau)$ has integral $O(t)$ as $t \to \infty \to$ risk of exponential unstability (not balanced by the damping)
- But it is also more and more concentrated on discrete times $\tau = kt/(k-\ell)$
- This is the effect of plasma echoes, discovered and experimentally observed in the sixties.
- Stabilizing role of the fact that the non-linearity answer of the plasma to itself is echoed (time localized) shown by our study

Stabilization by echoes (II)

VOLUME 19, NUMBER 5 PHYSICAL REVIEW LETTERS 31 JULY 1967

PLASMA WAVE ECHO*

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It is shown that if a longitudinal wave is excited in a collision-free plasma and Landau-damps away, and a second wave is excited and also damps away, then a third wave (i.e., the echo) will spontaneously appear in the plasma.

Stabilization by echoes (III)

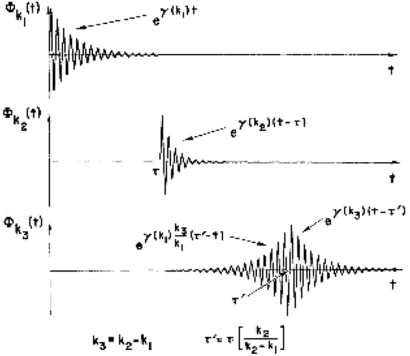

FIG. 1. Approximate variation of the principal Fourier coefficients of the self-consistent field for the case $k_3 \cong k_1 \cong \frac{1}{2}k_2$. Upper line: response to the first pulse; middle line: response to the second pulse; lower line: echo.

Stabilization by echoes (IV)

Heuristic:

$$y'(t) \leq A + \int_0^t y(\tau)\, K(t,\tau)\, d\tau$$

K has mass $O(t)$

If spread out, $K = \text{cst} = B$: exponential growth e^{BT}

If one "echo" (Dirac mass): $K = t\, \delta_{\theta t}$ then:
- $\theta = 0$: quadratic growth $O(t^2)$
- $\theta = 1$: no control
- $\theta = 1/2$: growth like $O(e^{C\sqrt{t}})$

Heuristic complicated by: infinitely many echoes, approaching the "bad" case $\theta = 1\ldots$

Stabilization by echoes (V)

- In the end, part of the gliding regularity of $\bar{f}$ has been converted into a large-time decay
- Case $\gamma > 1$ (Sobolev or even analytic regularity for ϕ): subexponential response of the type $e^{t^{1/\gamma}}$
- Case $\gamma = 1$ (Coulomb-Newton for ϕ): again subexponential growth by estimating separately each frequency and using the fact that echoes occuring at different frequencies are asymptotically well separated
- Then if rate of phase mixing stronger, instability can be controlled by an arbitrarily small loss of gliding regularity (price: gigantic constant, absorbed by the ultrafast convergence of the Newton scheme)

Comments & Perspectives

Many interesting open problems:

- Lin-Zeng obstruction to damping below $H^{3/2}$: huge gap with analyticity! Damping in higher Sobolev spaces?
- Inviscid damping for 2D incompressible Euler (damping around shear flows for instance)
- Stability and relaxation with a general confining geometry (cf. magnetic plasma)
- Self-gravitating galaxy models
- Stability of BGK waves in a plasma
- "Regularity transfer" structure in the phase space for other PDEs, e.g. Schrödinger with Wigner formulation?
- Links with weak KAM and weak turbulence?