Boundary regularity of solutions of degenerate elliptic equations without boundary conditions

Gary M. Lieberman

Iowa State University

November 15, 2011
1 Introduction

2 The oblique derivative problem

3 Application to degenerate equations

4 Connection to ODEs
If the linear second order elliptic operator $L$ is nondegenerate with smooth coefficients, then, for any smooth boundary values $\varphi$ on a smooth domain $\Omega \subset \mathbb{R}^n$, there is a smooth solution of the Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

assuming that $L1 \leq 0$. 
If $L$ is degenerate, however, then this problem may not be solvable for arbitrary smooth $\varphi$. Fichera developed a general theory of such problems which was further extended by Oleinik and Radkevich. In particular, no boundary condition need be prescribed on certain parts of the boundary, determined by the behavior of the coefficients of $L$ near an individual point on the boundary.
Here, we analyze a special situation in more detail. Write the operator as

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu$$

and suppose that $[a^{ij}]$ is a uniformly positive definite matrix, but the vector $b$ satisfies the condition that $|b|d$, where $d$ is distance to $\partial \Omega$, is bounded from below by a positive constant near $\partial \Omega$. 
In this case, we write \( b^i = \beta^i / d \) and \( \gamma = c / d \), and we assume that \( \beta \cdot Dd > 0 \) on the boundary. We consider the problem

\[
d \Delta u + \beta^i D_i u + \gamma u = \varphi \quad \text{in } \Omega.
\]

We also assume that \( \beta \) is bounded. (More general highest order terms may be considered.)
Langlais showed in 1984 that this problem has a unique solution for each smooth $\varphi$ and that the problem is smooth assuming that $\gamma$ is bounded from above by a sufficiently large negative constant. His method was to estimate solutions of the approximating boundary value problem

$$(d + \varepsilon)a^{ij}D_{ij}u + \beta^i D_i u + \gamma u = \varphi \text{ in } \Omega,$$

$$\beta^i D_i u + \gamma u = \varphi \text{ on } \partial\Omega$$

with a parameter $\varepsilon \to 0$. 
In fact, he only proved a bound on the second derivatives of $u$ and then quoted a general theorem which requires some heavy machinery due to Bolley and Camus. This machinery requires $\beta \cdot Dd > \frac{1}{2}$ near $\partial \Omega$. 
For other reasons, I was looking at the oblique derivative problem

\[ \Delta u = f \text{ in } \Omega, \quad \beta^i D_i u + \gamma u = \varphi \text{ on } \partial \Omega, \]

with \( f = O(d^{\alpha-1}) \) for some \( \alpha \in (0, 1) \), and I observed two things:

- \( v = \beta^i D_i u + \gamma u - \varphi \) (with \( \beta, \gamma, \varphi \) extended into \( \Omega \)) solves a Dirichlet problem, so one can estimate the Hölder norm \( |v|_\alpha \) rather easily.
- The estimate for \( v \) gives a Hölder estimate for \( Du \).
For our problem, by looking locally, we may assume that $\beta$ and $\varphi$ is constant and $\gamma \equiv 0$, and then $v$ satisfies the conditions

$$(d + \varepsilon)\Delta v + \beta^i D_i v - \frac{\beta \cdot Dd}{d + \varepsilon} v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega.$$ 

It’s not difficult to adapt previous arguments to get a uniform Hölder estimate for $v$, so $v = O(d^\alpha)$ and then use the equation

$$\Delta u = \frac{v}{d + \varepsilon}$$

to conclude that $\Delta u = O(d^{\alpha - 1})$. 
In fact, we can get Schauder type estimates, so our original problem has a solution in $C^{1, \alpha_0}$ for some $\alpha_0 \in (0, 1)$ (determined by $\Omega$) if $\Omega$ is a Lipschitz domain, and the solution is in $C^{k, \alpha}$ if the coefficients $a^{ij}$ are in $C^{k-2, \alpha}$, $\beta$ and $\gamma$ are in $C^{k-1, \alpha}$ and $\partial \Omega \in C^{k-1, \alpha}$ for any given $\alpha \in (0, 1)$. 
These ideas also show how the elliptic equation is like the solution of the ordinary differential equation

\[ u'' + \frac{a}{x}u' = 0 \]

near \( x = 0 \). If \( a \neq 1 \), the general solution is

\[ u = c_1 + c_2 x^{1-a} \]

so all solutions are globally continuous at \( x = 0 \) if \( a < 1 \). If \( a > 0 \), the only globally \( C^1 \) solutions are constants, and if \( a > 1 \), then the only bounded solutions are constants. The constant solutions are the unique solutions of the ODE with Neumann condition \( u'(0) = 0 \).
For the higher-dimensional (that is, PDE) case, we have a similar situation but nonzero Neumann-like data. This method shows that there is a unique $C^1$ solution if $\beta \cdot Dd > 0$ on $\partial \Omega$ and it’s as smooth as data allow. If $\beta \cdot Dd > 1$, then there is a unique bounded solution of the differential equation, and it’s as smooth as data allow. We also don’t need any restrictions on $\gamma$ (except for being negative).