Coherent Structures and Transport in Transitory Systems

Brock Mosovsky
James D. Meiss
University of Colorado, Boulder
Department of Applied Math

Theory and Computation of Lagrangian Coherent Structures, Part I
DS 11, Snowbird, UT
May 22, 2011
A Transitory Dynamical System with transition time $\tau$ satisfies

$$\dot{x} = V(x, t), \quad V(x, t) = \begin{cases} P(x) & \text{if } t < 0 \\ F(x) & \text{if } t > \tau \end{cases}$$
A **Transitory Dynamical System** with transition time $\tau$ satisfies

$$\dot{x} = V(x, t), \quad V(x, t) = \begin{cases} P(x) & \text{if } t < 0 \\ F(x) & \text{if } t > \tau \end{cases}$$

Modeled with convex combination of stationary vector fields:

$$V(x, t) = (1 - s(t))P(x) + s(t)F(x)$$

where

$$s(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > \tau \end{cases}$$
Definition

- A *Transitory Dynamical System* with transition time $\tau$ satisfies

$$\dot{x} = V(x, t), \quad V(x, t) = \begin{cases} P(x) & \text{if } t < 0 \\ F(x) & \text{if } t > \tau \end{cases}$$

- Modeled with convex combination of stationary vector fields:

$$V(x, t) = (1 - s(t))P(x) + s(t)F(x)$$

where

$$s(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > \tau \end{cases}$$

- e.g. $s(t) = t^2(3 - 2t)$
Applications

- Finite-time mixing

\[\text{Images courtesy of Wikipedia.org, dailygalaxy.com, and T. Shinbrot et al., Nature 397}\]
Applications

• Finite-time mixing

• Accelerator physics
Applications

- Finite-time mixing
- Accelerator physics
- Systems with multiple time scales (fast-slow dynamics) i.e. geophysical flows, relaxation oscillators, etc.
- *Any* system that transitions from one steady state to another (possibly the same) steady state over a finite time interval

---

1 Images courtesy of Wikipedia.org, dailygalaxy.com, and T. Shinbrot et al., Nature 397
Invariance and Past/Future Hyperbolicity

Can we rigorously quantify the flux between past and future coherent structures?
The Big Question:

Can we rigorously quantify the flux between past and future coherent structures?
We calculate areas of 2D lobes by computing:

- **Double integral**: \( \text{Area}(R) = \int \int_R dxdy \)

Action Difference (2D)
We calculate areas of 2D lobes by computing:

- Double integral: \( \text{Area}(R) = \int\int_{R} dxdy \)
- Contour integral (Stokes’ Theorem): \( \text{Area}(R) = - \oint_{\partial R} y \, dx \)
We calculate areas of 2D lobes by computing:

- **Double integral:** \( \text{Area}(R) = \iiint_R \, dx \, dy \)

- **Contour integral (Stokes’ Theorem):** \( \text{Area}(R) = - \oint_{\partial R} y \, dx \)

- **Action difference (main result)**

\[
\text{Area}(R) = \int_{-\infty}^{\infty} \left[ L(h_1(s), s) - L(h_2(s), s) \right] \, ds
\]

\[
L(x, y, t) = y \dot{x} - H(x, y, t)
\]
Action Difference (2D)

We calculate areas of 2D lobes by computing:

- Double integral: \( \text{Area}(R) = \iiint_R dx \, dy \)
- Contour integral (Stokes’ Theorem): \( \text{Area}(R) = - \oint_{\partial R} y \, dx \)
- Action difference (main result)

\[
\text{Area}(R) = \int_{-\infty}^{\infty} \left[ L(h_1(s), s) - L(h_2(s), s) \right] \, ds
\]

\( L(x, y, t) = y \dot{x} - H(x, y, t) \)

\( \Rightarrow \) Only need to know the two orbits \( h_1 \) and \( h_2 \)!
Example: Rotating Double Gyre

Streamfunction: \( \psi = -H = (1 - s(t))\psi_P(x) + s(t)\psi_F(x) \)

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial y}, \\
\dot{y} &= -\frac{\partial H}{\partial x}
\end{align*}
\]
Example: Rotating Double Gyre

Streamfunction: \( \psi = -H = (1 - s(t))\psi_P(x) + s(t)\psi_F(x) \)

\[
\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}
\]

\( \psi_P(x) = \sin(2\pi x) \sin(\pi y) \)
Example: Rotating Double Gyre

Streamfunction: \( \psi = -H = (1 - s(t))\psi_P(x) + s(t)\psi_F(x) \)

\[
\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}
\]

\( \psi_P(x) = \sin(2\pi x) \sin(\pi y) \)

\( \psi_F(x) = \sin(\pi x) \sin(2\pi y) \)
Example: Rotating Double Gyre

Streamfunction: \( \psi = -H = (1 - s(t))\psi_P(x) + s(t)\psi_F(x) \)

\[
\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}
\]

\( \psi_P(x) = \sin(2\pi x) \sin(\pi y) \)

\( \psi_F(x) = \sin(\pi x) \sin(2\pi y) \)

Contour Movie
- Quantify transport between coherent structures
- Flux from right gyre in $P$ to top gyre in $F$
- Compute $A_{rt}$ (dark blue).
Flux Computations: Rotating Double Gyre

\[ A_{rt} = - \int_{-\infty}^{\tau} \left[ L(h_1(s), s) - L(p_1(s), s) \right] ds - \frac{1}{2} (1 - (h_1)_x) - ([T(p_1)]_x - 1) \]
It can be shown that the transitory double gyre system is adiabatic, which in this case implies

$$A_{rt} \rightarrow 0.5 \quad \text{as} \quad \tau \rightarrow \infty.$$
Example: Transitory ABC Flow

\[
\begin{align*}
\dot{x} &= A \sin z + C \cos y , \\
\dot{y} &= A \cos z + B \sin x , \\
\dot{z} &= B \cos x + C \sin y .
\end{align*}
\]
Example: Transitory ABC Flow

\[ \dot{x} = A \sin z + s(t)C \cos y, \]
\[ \dot{y} = A \cos z + (1 - s(t))B \sin x, \]
\[ \dot{z} = (1 - s(t))B \cos x + s(t)C \sin y. \]
Example: Transitory ABC Flow

\[ \dot{x} = A \sin z + s(t)C \cos y , \]
\[ \dot{y} = A \cos z + (1 - s(t))B \sin x , \]
\[ \dot{z} = (1 - s(t))B \cos x + s(t)C \sin y. \]

\[ P(x, z) \quad (C = 0, \ H_P = \dot{y}) \]
Example: Transitory ABC Flow

\[ \dot{x} = A \sin z + s(t)C \cos y, \]
\[ \dot{y} = A \cos z + (1 - s(t))B \sin x, \]
\[ \dot{z} = (1 - s(t))B \cos x + s(t)C \sin y. \]

\[ P(x, z) \quad (C = 0, \ H_P = \dot{y}) \]

\[ F(y, z) \quad (B = 0, \ H_F = \dot{x}) \]
Example: Transitory ABC Flow

Lobe Movie
Example: Transitory ABC Flow

Lobe Movie

Goal: Quantify transport between past and future resonances.

\[ P(x, z) \quad (C = 0) \hspace{2cm} F(y, z) \quad (B = 0) \]
Heteroclinic Intersections

- 3D lobes $\Rightarrow$ 2D manifolds $\Rightarrow$ 1D intersections (curves)
Heteroclinic Intersections

- 3D lobes $\Rightarrow$ 2D manifolds $\Rightarrow$ 1D intersections (curves)
- Must compute intersection curves to apply flux formulas
Heteroclinic Intersections

- 3D lobes $\Rightarrow$ 2D manifolds $\Rightarrow$ 1D intersections (curves)
- Must compute intersection curves to apply flux formulas

Intersection Curve Movie
Flux Results: Lobe Volumes

Primary

Secondary
Conclusions

- Quantify flux in exact volume-preserving transitory flows
Conclusions

- Quantify flux in exact volume-preserving transitory flows
- Requires little Lagrangian information
Conclusions

• Quantify flux in exact volume-preserving transitory flows
• Requires little Lagrangian information
• Both analytical and numerical results
Conclusions

- Quantify flux in exact volume-preserving transitory flows
- Requires little Lagrangian information
- Both analytical and numerical results
- Provides insight into more general nonautonomous case
Acknowledgments

Thanks:
SIAM and the organizers for hosting this minisymposium.

Publication:

Support:
Funding for B. Mosovsky and J. D. Meiss provided by NSF Grant #DMS-0707659 and the MCTP Program at CU Boulder
Proof of 2D Action-Flux Formulas

For phase space Lagrangian \( L(x, y, t) \), we have

\[
\frac{d}{dt} \nu = \mathcal{L}_V \nu = \nu (d\nu) + d(\nu V) = d(y\dot{x} - H) = dL.
\]

Also

\[
\nu - \varphi^*_{t_1, t_2} \nu = \int_{t_1}^{t_2} \frac{d}{ds} \varphi^*_{s, t_2} \nu \, ds = \int_{t_1}^{t_2} d(\varphi^*_{s, t_2} L) \, ds.
\]

Then for the integral over an \textit{unstable} segment,

\[
\int_{\mathcal{U}_{t_2}} \nu = \int_{\mathcal{U}_{t_2}} \left( \int_{t_1}^{t_2} d(\varphi^*_{s, t_2} L) \, ds + \varphi^*_{s, t_2} \nu \right)
\]

\[
= \int_{t_1}^{t_2} \left( \int_{\mathcal{U}_s} dL \right) \, ds + \int_{\mathcal{U}_{t_1}} \nu
\]

\[
= \int_{t_1}^{t_2} L(h_2(s), s) - L(h_1(s), s) \, ds + \int_{\mathcal{U}_{t_1}} \nu
\]

\[
= \int_{-\infty}^{t_2} L(h_2(s), s) - L(h_1(s), s) \, ds
\]
We generalize the preceding result. For exact volume-form $\Omega$, let

$$d\alpha = \Omega, \quad \frac{d}{dt}\alpha = \mathcal{L}_V \alpha = d\lambda.$$  

- Let $\mathcal{U}_\tau \subset W_\tau^u(\gamma(\partial \mathcal{U}, \tau))$ and $|\partial \mathcal{U}_t| \to 0$ as $t \to -\infty$. Then

$$\int_{\mathcal{U}_\tau} \alpha = \int_{-\infty}^\tau \left( \int_{\partial \mathcal{U}_s} \lambda \right) ds.$$  

- Let $\mathcal{S}_\tau \subset W_\tau^s(\gamma(\partial \mathcal{S}, \tau))$ and $|\partial \mathcal{S}_t| \to 0$ as $t \to \infty$. Then

$$\int_{\mathcal{S}_\tau} \alpha = -\int_{\tau}^\infty \left( \int_{\partial \mathcal{S}_s} \lambda \right) ds.$$
Past and Future Hyperbolicity

- Traditional notion of hyperbolicity is too strong.
- Define backward/forward hyperbolicity for transitory systems in terms of “half-time” information.
- An invariant set $\Lambda_{t_0}$ at time $t = t_0$ is
  - *backward hyperbolic* if $\varphi_{t,t_0}(\Lambda_{t_0})$ is hyperbolic under $P(x)$ for all $t \leq 0$.
  - *forward hyperbolic* if $\varphi_{t,t_0}(\Lambda_{t_0})$ is hyperbolic under $F(x)$ for all $t \geq \tau$. 
Motivation

- Invariant manifolds organize phase space.
- Can be complicated for time-dependent systems.
- “Turnstile” mechanism responsible for transport.

Perturbed Pendulum Movie