Mean Field Stackelberg Games: Coalescence of Hysteresis

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Overview

1. Introduction
   - Mean Field Games
   - Different Population

2. Coalescence of Hysteresis in a Large Population
   - Problem Setting
   - $\epsilon$-Nash Equilibrium
   - Linear Quadratic Case

3. Conclusion
- Model the interactive behavior of a group of decision makers (agents)
- Complexity accelerates with the number of participants
- Stochastic Differential Games (SDGs)
  Specify individuals’ evolutionary dynamics and objectives, see Bensoussan and Freshe [2000,2009] for non-zero sum SDGs with N players.
Huang et al. [2003] investigated SDG problems involving infinitely many players under the name “Large Population Stochastic Dynamic Games”.

Independently, Lasry and Lions [2007] studied similar problems from the viewpoint of the mean-field theory in physics and termed “Mean-Field Games (MFGs)”.

Under the mean field framework, instead of highlighting the interaction between any two agents explicitly; each individual now interacts with a common medium created in the community: the mean field term.

Mathematically, the mean field term is a functional of the probability distribution of the population.
The theory of MFGs has 3 crucial assumptions:

1. A continuum of homogeneous players
2. Homogeneity in strategic performance
3. Social interactions through the mean field term

In the contemporary MFGs, each agent considers the mean field term as exogenous. That is, each individual’s decision has infinitesimal effect on the whole community.

MFG problems naturally possess the forward-backward structure.
To generalize the classical MFG, one can consider non-homogeneous populations. In particular, Huang [2010]

- MFG with a big player under LQ setting.
- The mean field term (conditional expectation of the small players) is **Exogenous** to the whole problem, for both the big and small players.

Nourian and Caines [2013]

- MFG with a big player under a general setting.
- The mean field term (conditional probability measure of the small players) is again **Exogenous**.

These setting are rare in economic world and policy making.
- From the viewpoint of policy making, it's natural to assume that the governor has the power of giving direct influence on the whole community.

- Consider the mean field term as **Endogenous** to the dominating player (Stackelberg Player):
  
  i.e. the mean field term is substantially guided and controlled by the "big" player (not a major player).

- In the presence of a dominating player, we assume naturally that there is a technological limitation to each agent so that he can only grasp the information coming from the dominating player at a delayed time.

- In the literature for *delay* problems in stochastic controls or MFGs, they usually refer to the delay in their own states or controls of agents. In our problem formulation, the delay information (hysteresis) is generated by the third party: the dominating player.
Each agent now solves for his own optimal control problem based on three sources of information flows, namely:

1) Agent’s own noise;
2) Delayed information (hysteresis) coming from the dominating player;
3) The aggregated (coalesced) public information via the mean field term.

- The filtration generated by the third source through the mean field term carries extra information.
- Other agents with smaller size of delay would also contribute to the aggregated (coalesced) information.
Problem Setting: Dominating player and hysteresis

Assume that the initial path, \( \{ \xi_0(t) : t \in [-b, 0] \} \) is square integrable and the initial random variables \( \{ \xi'_1 \}_{i \in \{1, \ldots, N\}} \) are square integrable, identically and independently distributed, all of them are also independent with \( \xi_0 \).

Define the following filtrations,

\[
\mathcal{F}_t^0 = \begin{cases} 
\sigma(\xi_0(s) : s \leq t), & t \in [-b, 0]; \\
\sigma(\xi_0, W_0(s) : s \leq t), & t > 0;
\end{cases}
\]

\[
\mathcal{F}_t^{1,i} = \sigma(\xi_1, W_1^i(s) : s \leq t), & t > 0.
\]
The empirical state evolutions of the dominating player and the $i^{th}$ player (with hysteresis $\delta_i$) are respectively described by $y_0$ and $y_{1,i,\delta_i}^{\ast} = y_{1,i,\delta_i}$, which satisfy the SDEs

$$
\begin{align*}
\frac{dy_0}{dt} &= g_0(y_0(t), \frac{\sum_{j=1}^{N} y_{1,j}^{\ast,\Delta_j}(t)}{N}, u_0(t))dt + \sigma_0 \, dW_0(t) \\
y_0(t) &= \xi_0(t), \quad t \in [-b, 0]; \\
\frac{dy_{1,i,\delta_i}}{dt} &= g_1\left(y_{1,i,\delta_i}(t), \frac{\sum_{j=1, j \neq i}^{N} y_{1,j}^{\ast,\Delta_j}(t)}{N-1}, u_{1,i,\delta_i}(t), y_0(t-\delta_i)\right)dt + \sigma_1 \, dW_1(t) \\
y_{1,i,\delta_i}(0) &= \xi_i.
\end{align*}
$$

Form the $i^{th}$ player's perspective, $\{\Delta_j\}_{j \in \{1, \ldots, i-1, i+1, \ldots, N\}}$ is a sequence of independent and identically distributed random variables on $\mathbb{R}$, where $\Delta_j$ represents the hysteresis (delay) parameter for the $j^{th}$ player.
Assumptions

A.1 Lipschitz Continuity

$g_0$ and $g_1$ are globally Lipschitz continuous in all arguments, i.e. there exists $L > 0$, such that

$$|g_0(x_0, z, v_0) - g_0(x'_0, z', v'_0)| \leq L(|x_0 - x'_0| + |z - z'| + |v_0 - v'_0|);$$

$$|g_1(x_1, z, v_1, x_0) - g_1(x'_1, z', v'_1, x_0)| \leq L(|x_1 - x'_1| + |z - z'| + |v_1 - v'_1| + |x_0 - x'_0|).$$
A.2 Linear Growth

g₀ and g₁ are of linear growth in all arguments, i.e. there exists \( L > 0 \), such that

\[
|g₀(x₀, z, v₀)| \leq L(1 + |x₀| + |z| + |v₀|);
\]

\[
|g₁(x₁, z, v₁, x₀)| \leq L(1 + |x₁| + |z| + |v₁| + |x₀|).
\]

A.3 Quadratic Condition (See (A.5) in Carmona and Delarue [3].)

There exists \( L > 0 \), such that

\[
|f₁(x₁, z, v₁, x₀) - f₁(x₁', z', v₁', x₀')| \leq L \left[ 1 + |x₁| + |x₁'| + |z| + |z'| + |v₁| + |v₁'| + |x₀| + |x₀'| \right] \cdot \left[ |x₁ - x₁'| + |z - z'| + |v₁ - v₁'| + |x₀ - x₀'| \right].
\]
Assume that each participant together with the dominating player has the knowledge of the prior probability measure of $\Delta$, which is denoted by $\pi_\Delta$.

Each $i^{th}$ player however only knows the magnitude of his own delay, but not that of the others. This is similar as in adverse selection markets.

In particular, we set $\Delta \in [a, b]$, where $a, b$ are some fixed finite positive numbers. We focus on the $i^{th}$ player, with his delay $\Delta_i = \delta_i$. Let $u = (u_1^{1, \delta_1}, u_2^{2, \delta_2}, \ldots, u_N^{N, \delta_N})$. The objective of the $i^{th}$ player is to minimize the cost functional:

$$J^{i, \delta_i, N}(u) = \mathbb{E} \int_0^T f_1(y_1^{i, \delta_i}(t), \frac{\sum_{j=1, j \neq i}^N y_1^{j, \Delta_j}(t)}{N-1}, u_1^{i, \delta_i}(t), x_0(t-\delta_i)) \, dt.$$
Let $x_0$ and $x_{1,\delta_i}$ be the mean-field analogy satisfying the SDEs:

$$
\begin{align*}
    dx_0 &= g_0(x_0(t), z(t), u_0(t))dt + \sigma_0 dW_0(t) \\
    x_0(t) &= \xi_0(t), \quad t \in [-b, 0]; \\
    dx_{1,\delta_i} &= g_1(x_{1,\delta_i}(t), z(t), u_{1,\delta_i}(t), x_0(t - \delta_i))dt + \sigma_1 dW_1(t) \\
    x_{1,\delta_i}(0) &= \xi_1.
\end{align*}
$$

The analogical cost functional for the $i^{th}$ player is given by

$$
J^{i,\delta_i}(u_{1,\delta_i}) = \mathbb{E} \int_0^T f_1\left(x_{1,\delta_i}(t), z(t), u_{1,\delta_i}(t), x_0(t - \delta_i)\right)dt.
$$
In the following, we can pick $z$ to be a process adapted to $\mathcal{F}_{t-a}^0$ such that $y_{i,\delta_i}^1$ converges to $x_{i,\delta_i}^1$.

The mean field filtration is defined by: $\mathcal{F}_t^z := \sigma(z(s) : s \leq t)$.

At time $t$, the $i^{th}$ player makes his decision based on:
- $\mathcal{F}_{t,i}^{1}.$: The $i^{th}$ player's own noise;
- $\mathcal{F}_{t-\delta_i}^0.$: The delayed information from the dominating player;
- $\mathcal{F}_t^z.$: The public information from the population through $z$.

It is natural to assume that $u^i(t)$ is $G_t^i := \mathcal{F}^{1,i}_t \vee \mathcal{F}^0_{t-\delta_i} \vee \mathcal{F}_t^z$-adapted.
We can establish the convergence of \( y_0 \rightarrow x_0 \) and \( y_1^{i, \delta_i} \rightarrow x_1^{i, \delta_i} \).

**Lemma 2.1 (Fixed point equilibrium)**

Suppose all agents adopt the optimal controls under the mean field system. Let \( \pi_\Delta \) be any probability measure on \([a, b]\). If the mean-field term satisfies

\[
    z(s) = \int_{[a, b]} \mathbb{E}^{\mathcal{F}_s^{0, \delta}} \int_{\mathcal{F}_s^{2, \delta}} x_1^{i, \delta}(s) d\pi_\Delta(\delta),
\]

then

\[
    \mathbb{E} \left[ \sup_{t \leq T} |y_0(t) - x_0(t)|^2 \right] + \mathbb{E} \left[ \sup_{\delta \in [a, b]} \sup_{t \leq T} |y_1^{i, \delta}(t) - x_1^{i, \delta}(t)|^2 \right] \rightarrow 0.
\]

The convergence for the cost functional is similar.
We can conclude with

**Theorem 2.2**

\[ u = (u_1^1, u_2^1, \ldots, u_N^1); \] where \{u_i^1 : i \geq 1\} are optimal with respect to the mean field control problem, is an \( \epsilon \)-Nash equilibrium. In particular, we have

\[ \mathcal{J}^{i, \delta_i, N}(u) \leq \mathcal{J}^{i, \delta_i, N}(v) + o(1), \]

where \( v = (u_1^1, \ldots, u_{i-1}^1, v_i, u_{i+1}^1, \ldots, u_N^1) \).

We note that the convergence rate is different, only topological one, unlike those commonly found in the literature of \( O\left(\frac{1}{\sqrt{N}}\right) \).
We illustrate our theory through a linear quadratic control problem as follows:

\[
\begin{align*}
    d\mathbf{x}_0 &= \left( A_0 \mathbf{x}_0(t) + B_0 \mathbf{z}(t) + C_0 \mathbf{u}_0(t) \right) dt + \sigma_0 dW_0(t) \\
    \mathbf{x}_0(t) &= \xi_0(t), \quad t \in [-b, 0]; \\
    d\mathbf{x}_1^{\delta} &= \left( A_1 \mathbf{x}_1^{\delta}(t) + B_1 \mathbf{z}(t) + C_1 \mathbf{u}_1(t) + D \mathbf{x}_0(t - \delta) \right) dt + \sigma_1 dW_1(t) \\
    \mathbf{x}_1^{\delta}(0) &= \xi_1,
\end{align*}
\]

where the mean field term \( \mathbf{z}(t) \) is to be defined later, whose functional form is adapted to \( \mathcal{F}^{0}_{-a} \) and hence is directly influenced by the dominating player. The control for the dominating player, \( \mathbf{u}_0(t) \), is \( \mathcal{F}^0_t \) adapted; while the control for the representative agent, \( \mathbf{u}_1^{\delta}(t) \) is \( \mathcal{G}^{\delta}_t = \mathcal{F}^1_t \cup \mathcal{F}^0_{t-\delta} \cup \mathcal{F}^2_t \) adapted.
We denote $M^T$ the transpose of any matrix $M$. Suppose $Q_i, R_i > 0$; $i = 0, 1$, we consider the following problems:

**Problem 2.3**

*Given the process $x_0$ and $z$, find a control $u_1^\delta$ which minimizes the cost functional:*

$$J_1(u_1^\delta, x_0, z) = \mathbb{E} \left[ \int_0^T \left| x_1^\delta(t) - E_1 z(t) - F x_0(t - \delta) - G_1 \right|^2_{Q_1} + \langle u_1^\delta(t), R_1 u_1^\delta(t) \rangle dt \right].$$
Problem 2.4

Find the process $z$ such that

$$z(t) = \int_{[a,b]} \mathbb{E}^{F_{t-\delta} \vee F_t} x_1^\delta(t) d\pi_\Delta(\delta),$$

where $x_1^\delta$ is the controlled SDE using $u_1^\delta$ solved by Problem 2.3.

Problem 2.5

Find a control $u_0$ which minimizes the cost functional

$$J_0(u_0) = \mathbb{E} \left[ \int_0^T \left| x_0(t) - E_0 z(t) - G_0 \right|_{Q_0}^2 + \left\langle u_0(t), R_0 u_0(t) \right\rangle dt \right],$$

where $z$ is the solution given in Problem 2.4.
Observe that the representative agent's decision at time $t$ is $\mathcal{G}_t^\delta$ adapted.

- If $\delta = a$, then $\mathcal{G}_t^a = \mathcal{F}_t^1 \lor \mathcal{F}_{t-a}^0 \lor \mathcal{F}_{t}^{z} = \mathcal{F}_t^1 \lor \mathcal{F}_{t-a}^0$, as $\mathcal{F}_{t}^{z}$ is a sub $\sigma$-algebra of $\mathcal{F}_{t-a}^0$, is a Brownian filtration.

- Otherwise, in general for $a \neq b$, $\mathcal{G}_t^\delta$ is not necessarily a Brownian one.

The classical FBSDE solved with Martingale Representation Theorem (MRT) is not sufficient without a Brownian filtration. Inspired by the ideas in G. Liang, T. Lyons and Z. Qian. [2011], we can work on forward backward dynamics on a non-Brownian filtration. To motivate our further development, we here provide a brief introduction to Backward Dynamics.
In particular, we want to solve for \((y_t, M_y(t))\) satisfying a stochastic backward equation on an arbitrary filtration \(\mathcal{H}_t\)

\[ y_t = \xi + \int_t^T g(y_s) ds - \int_t^T dM_y(s), \]

or in differential form

\[ dy_t = -g(y_t) dt + dM_y(t), \]

where \(y\) is square integrable, \(\xi\) is the terminal random variable and \(M_y\) is an \(\mathcal{H}\)-martingale. We assume that the generator \(g\) satisfies some regularity assumptions (for example, global Lipschitz and linear growth) to guarantee the unique existence of an adapted solution.
Taking conditional expectation on the backward dynamics

\[ y_t = \xi + \int_t^T g(y_s) \, ds - \int_t^T dM_y(s), \]

we have

\[ y_t = \mathbb{E}^\mathcal{H}_t \left[ \xi + \int_0^T g(y_s) \, ds \right] - \int_t^0 g(y_s) \, ds \]

or in differential form

\[ dy_t = -g(y_t) \, dt + d\mathbb{E}^\mathcal{H}_t \left[ \xi + \int_0^T g(y_s) \, ds \right]. \]

Note that \( \mathbb{E}^\mathcal{H}_t \left[ \xi + \int_0^T g(y_s) \, ds \right] \) is clearly a \( \mathcal{H} \)-martingale, and hence

\[ M_y(t) := \mathbb{E}^\mathcal{H}_t \left[ \xi + \int_0^T g(y_s) \, ds \right]. \]
Further, if we define $V(y)_t := \int_0^t g(y_s)ds$, then solving the backward dynamics is equivalent to tackle the fixed point problem of

$$y_t = \mathbb{E}^{H_t} \left[ \xi + V(y)_T \right] - V(y)_t,$$

which gets rid of the usage of MRT. Throughout this paper, for any backward equation $y$, we refer $M_y$ to be the martingale defined by its terminal and generator as in

$$M_y(t) := \mathbb{E}^{H_t} \left[ \xi + \int_0^T g(y_s)ds \right].$$
We first solve for the control problem for the representative agent.

**Lemma 2.6**

**Control for the Representative Agent**

Problem 2.3 is uniquely solvable and the optimal control is $-R_1^{-1}C_1^T n^\delta(t)$, such that $n^\delta$ satisfies the backward dynamics:

$$-dn^\delta = \left( A_1^T n^\delta(t) + Q_1(x_1^\delta(t) - E_1 z(t) - Fx_0(t - \delta) - G_1) \right) dt$$

$$-dM_{n^\delta}(t)$$

$$n^\delta(T) = 0.$$
To obtain the mean field equilibrium stated in Problem 2.4

\[ z(t) = \int_{[a,b]} \mathbb{E}_{\mathcal{F}_{t-\delta}^0 \vee \mathcal{F}_t^z} x_1^\delta(t) d\pi_\Delta(\delta), \]

we take expectation conditional on \( \mathcal{F}_{t-\delta}^0 \vee \mathcal{F}_t^z \) and integrate on \( \delta \) over \([a, b]\) on both sides of \((x_1^\delta, n^\delta)\), which yields

\[
\begin{align*}
    dz &= (A_1 + B_1)z(t) - C_1 R_1^{-1} C_1^T m(t) + D \int_{[a,b]} x_0(t-\delta) d\pi_\Delta(\delta) \, dt \\
    z(0) &= \mathbb{E}[\xi_1] \\
    -dm &= (A_1^T m(t) - Q_1 F \int_{[a,b]} x_0(t-\delta) d\pi_\Delta(\delta) \\
    &\quad + Q_1(I - E_1)z(t) - Q_1 G_1) \, dt - dM_m(t) \\
    m(T) &= 0.
\end{align*}
\]

where we write \( m(t) = \int_{[a,b]} \mathbb{E}_{\mathcal{F}_{t-\delta}^0 \vee \mathcal{F}_t^z} n^\delta(t) d\pi_\Delta(\delta). \)
$(z, m)$ are clearly adapted to $F_{t-a}^0$ adapted, and hence $(z, m)$ are ordinary FBSDE (adapted to the Brownian Filtration $F_{t-a}^0$). Indeed, by MRT, $dM_m(t)$ has to be $Z(t)dW(t-a)$ for some $Z(t) \in F_{t-a}^0$.

Before proceeding to Problem 2.5, we first discuss the unique existence of the system $(z, m)$, which is due to the affine construction:

**Lemma 2.7**

*Given any square integrable process $x_0$, suppose the non-symmetric Riccati equation*

$$
\frac{d\Gamma}{dt} + \Gamma_t(A_1 + B_1) + A_1^T\Gamma_t - \Gamma_t C_1 R_1^{-1} C_1^T \Gamma_t + Q_1(I - E_1) = 0,
$$

$$
\Gamma(T) = 0,
$$

*admits a unique solution on $[0, T]$, then there uniquely exists a solution to $(z, m)$.*
Using Theorem III.5 in Bensoussan et al. [2011], we have the following proposition.

**Proposition 2.8**

**Case 1:**
If \( n_1 = 1 \), then \( \Gamma \) always admits a solution on \([0, T]\).

**Case 2:**
If \( n_1 > 1 \), suppose that there is a representation \( Q_1(l - E_1) = Q + S \), where \( Q \) is positive definite, such that

\[
\left(1 + \sqrt{T}\|e^{A_1^T}\|BQ^{-\frac{1}{2}}\right)\left(1 + \|Q^{-\frac{1}{2}}SQ^{-\frac{1}{2}}\|\right) < 2,
\]

then \( \Gamma \) admits a unique solution on \([0, T]\). Here,

\[
\|e^{A_1^T}\| = \sup_{t \leq T} \sqrt{\int_t^T |e^{A_1^T(s-t)}Q_{1/2}|^2 \, ds}.
\]
We next turn to the control problem for the dominating player. Note that we can decompose the system into \((z_0, m_0) \in \mathcal{F}_{-a}^0\) and a deterministic component \((z_c, m_c)\), such that \((z, m) = (z_0 + z_c, m_0 + m_c)\):

\[
\begin{align*}
\frac{dz_0}{dt} &= (A_1 + B_1)z_0(t) - C_1R_1^{-1}C_1^Tm_0(t) \\
&\quad + D \int_{[a,b]} x_0(t - \delta) d\pi_\Delta(\delta) \, dt, \quad z_0(0) = 0; \\
-dm_0 &= (A_1^Tm_0(t) - Q_1F \int_{[a,b]} x_0(t - \delta) d\pi_\Delta(\delta) \\
&\quad + Q_1(1 - E_1)z_0(t)) \, dt - dM_m_0(t), \quad m_0(T) = 0.
\end{align*}
\]

We have \(x_0 \mapsto (z_0(x_0), m_0(x_0)) = (z_0, m_0)\) is linear. We consider the linear functional \(\mathcal{L} : L^2_{\mathcal{F}_0^0}([-b, T]; \mathbb{R}^{m_0}) \to L^2_{\mathcal{F}_a^0}([0, T]; \mathbb{R}^{m_1})\) defined by

\[
\mathcal{L}(x_0)(t) = z_0(t).
\]
It can be shown that $\mathcal{L}$ is bounded. By the Riesz Representation Theorem, the Hermitian adjoint operator $\mathcal{L}^*: L^2_{\mathcal{F}_{-b}}([0, T]; \mathbb{R}^{n_1}) \to L^2_{\mathcal{F}_0}([-b, T]; \mathbb{R}^{n_0})$ uniquely exists such that
\[
\mathbb{E} \int_0^T \langle f(t), \mathcal{L}(g)(t) \rangle dt = \mathbb{E} \int_{-b}^T \langle \mathcal{L}^*(f)(t), g(t) \rangle dt,
\]
for all $f \in L^2_{\mathcal{F}_{-b}}([0, T]; \mathbb{R}^{n_1})$, $g \in L^2_{\mathcal{F}_0}([-b, T]; \mathbb{R}^{n_0})$. In particular, we have $\|\mathcal{L}\| = \|\mathcal{L}^*\|$. 
The dynamics and the cost functional for the dominating player can be rewritten as
\[
\begin{align*}
dx_0 &= \left( A_0 x_0(t) + B_0 (\mathcal{L}(x_0)(t) + z_c(t)) + C_0 u_0(t) \right) dt + \sigma_0 dW_0(t) \\
x_0(t) &= \xi_0(t), \quad t \in [-b, 0].
\end{align*}
\]
and
\[
J_0(u_0) = \mathbb{E} \left[ \int_0^T \left( \left| x_0(t) - E_0 (\mathcal{L}(x_0)(t) + z_c(t)) - G_0 \right|_{Q_0}^2 \\
+ \left\langle u_0(t), R_0 u_0(t) \right\rangle dt \right] .
\]
Theorem 2.9

Control for the Dominating Player

The Dominating player’s optimal control is given by $-R_0^{-1} C_0^T p(t)$, where $p$ satisfies the functional BSDE

$$-dp = \left( A_0^T p(t) + \mathcal{L}^*(B_0^T p)(t) \\
+ Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0) \\
- \mathcal{L}^* \left( E_0^T Q_0(x_0 - E_0(\mathcal{L}(x_0) + z_c) - G_0) \right)(t) \right) dt - dM_p(t)$$

$p(T) = 0$. 
Theorem 2.10

Suppose that for any given $x_0$, the equilibrium pair $(z, m)$ always admits a unique solution, then the FBSFDE

\[
\begin{align*}
    dx_0 &= \left( A_0 x_0(t) + B_0 (L(x_0)(t) + z_c(t)) - C_0 R_0^{-1} C_0^T p(t) \right) dt + \sigma_0 dW_0(t), \\
    -dp &= \left( A_0^T p(t) + L^* \left( B_0^T p - E_0^T Q_0 (x_0 - E_0 (L(x_0) + z_c) - G_0) \right) \right)(t) \\
    &+ Q_0 \left( x_0(t) - E_0 (L(x_0)(t) + z_c(t)) - G_0 \right) dt - dM_p(t).
\end{align*}
\]

admits a unique solution.

Sketch of proof:
One can check that the FBSFDE satisfies certain monotonicity conditions. The result follows by an extension of the method of continuation.
Corollary 2.11

If \( n_1 > 1 \), suppose \( Q_1(I - E_1) = Q + S \), where \( Q \) is positive definite, such that

\[
\left( 1 + \sqrt{T} \| e^{A_T} \| \| BQ^{-\frac{1}{2}} \| \right) \left( 1 + \| Q^{-\frac{1}{2}} S Q^{-\frac{1}{2}} \| \right) < 2,
\]

then the non-symmetric Riccati equation \( \Gamma \) admits a unique solution on \([0, T] \). The equilibrium pair \((z, m)\), and hence the Mean Field Stackelberg Game (MFSTG), is uniquely solvable.

For the \( n_1 = 1 \), the MFSTG is always uniquely solvable.
- We study MFG with a Dominating player (Stackelberg Player), the mean field term being **Endogenous** to the Dominating player.
- We study a more practical case by introducing the Hysteresis effect of each agent.
  - Non-canonical fixed point property, convergence rate not common in MFG literature.
  - New tools have to be involved, e.g. Backward Dynamics, Functional FBSDEs.
- Recall that Huang[2010] consider the mean field term as exogenous, which reduces to our case by setting \( q \equiv 0 \equiv r \) and \( \Delta \equiv 0 \).
- More general setting to be introduced later in Mean Field Stackelberg Games with Tribal Heterogeneity, with (Nourian and Caines [2013]) as a special case.


Thank you!

Mean Field Games and Mean Field Type Control Theory

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